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New Insights from the Canonical Fisheries Model

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Sammendrag: Vi analyserer den kanoniske fiskerimodellen ved hjelp av optimal kontroll teori og utleder noen nye resultater. Vi viser at så lenge det ikke er optimalt å la fiskebestanden dø ut, vil det alltid være et intervall med lave bestander hvor det er optimalt å ikke høste. I motsetningen til litteraturen er dette resultatet er ikke avhengig av en antagelse om at marginal høstningskostnad øker med redusert fiskebestand. Resultatet blir brukt til å bevise at svake forhold skyggeprisen på fiskebestanden alltid går til uendelig når bestanden nærmer seg null. Resultatene generaliseres til aldersstrukturerte fiskerimodeller.

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New Insights from the Canonical Fisheries Model

Eric Nævdal[*] and Anders Skonhoft[§]

Abstract

We analyse the standard optimal control fishery model and derive some novel results. We show that as long as it is not optimal to let the stock become extinct, there will always be an interval with low stock sizes where it is optimal not to harvest. This result does depend on any assumption that marginal harvesting cost per unit increases with decreasing stock size. This result is then used to prove that weak conditions the shadow price on the fish stock always goes to infinity as the stock approaches zero. The results are generalized to age structured models.

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1. Introduction

Clark (1973) and Clark and Munro (1975) presented dynamic fishery models that gave the theory of renewable resources a proper capital theoretic foundation. The basic fishery model entails one control variable, one state variable; the planning horizon is infinite time and the problem is autonomous. When the profit function is nonlinear in the control variable and there is an optimal path to the steady state, this steady state should be approached gradually along two saddle paths, or stable manifolds (see, e.g., Kamien and Schwarz 1991). The standard model has usually applied a ecological lumped parameter model of the form $\dot{x} = G(x) - h$ where x is biomass and h is the harvest rate. It has been recognized for a long time that optimal extinction in these models depends on the relative magnitude of the interest rate and the intrinsic growth rate, $G'(0)$ in addition to the unit cost of harvesting, Clark (1973), Cropper (1979). Although this model is well understood, some wrinkles remain to be ironed out. One is the question of harvest levels at low stock levels. It is has been known for a long time that in the standard fisheries model it is optimal to set harvest equal to zero for low stock levels, Leung and Wang (1976), Lewis and Schmalensee (1977). This is commonly attributed to an assumption that harvests costs are stock dependent and that the marginal cost of harvest becomes very large or infinite when the stock goes to zero. We show below that this is not a necessary condition. In order to properly analyse optimal harvest levels at low stocks, it is crucial to examine the behaviour of the shadow price at low stock levels. We argue below that analysing the properties of the shadow price at low stock levels is equivalent to analysing the stable saddle path in a phase diagram in stock/shadow price space. We then show that the shadow price of a renewable resource goes to zero if the growth in the resource is zero at zero stock. This fact has remarkably not been noted in the literature. Colin Clark in his milestone book on natural resource economics is silent on this. He draws the basic fishery model phase-diagram in the stock – harvest space, but the saddle path illustration is not finalized when the harvest becomes low (Clark 1990, p. 99). See also Conrad and Clark (1987, p. 56). In the well-recognized book by Leonard and Long on optimization and dynamic control models, the saddle path illustrating a schooling fishery is only indicated for a restricted set of values in the stock – shadow price space (Leonard and Long (1992, p. 296) and is not drawn for values of the stock close to zero...

In what follows, we first formulate and analyse our baseline model exemplified by a schooling fishery with a fixed harvest price. We apply fast/slow-dynamics and show that the results apply to at least some age structured models.

2. The baseline model

The following is the basic version of the fisheries model where a schooling fishery is considered. In a schooling fishery there are no stock dependent harvest costs, and with a fixed fishing price p the problem is accordingly:

$$\max_{h \geq 0} \int_0^\infty \left(ph - C(h) \right) e^{-\rho t} dt \quad \text{subject to } \dot{x} = G(x) - h, \text{and } x(0) \text{ given}, \tag{1}$$

and where $h \geq 0$ is the harvest and $x > 0$ is the size of the fish stock, and $\rho \geq 0$ is the discount rate. The natural growth function $G(x)$ is assumed to be strictly concave and satisfy $G(0) = 0$, $G'(x) > 0$ over some interval $[0, \bar{x}$) and $G'(x) < 0$ for $x > \bar{x}$. It is also assumed that there is some number $K > \bar{x}$ such that $G(K) = 0$. These assumptions are in line with the standard logistic growth function, which is used in our numerical illustrations. The cost function $C(h)$ is assumed to be increasing, convex and satisfy $C'(0)=0$. For notational convenience we denote $C'(h)$ as $c(h)$.

The current value Hamiltonian for this problem is:

$$H = ph - C(h) + \mu\big(G(x) - h\big). \tag{2}$$

The Hamiltonian is concave in (h, x), so sufficiency theorems such as Theorem 9.11.1 in Sydsæter et al. (2005) are fulfilled. The necessary conditions become:

$$\frac{\partial H}{\partial h} = p - c(h) - \mu \le 0 \; (= 0 \text{ if } h > 0) \tag{3}$$

and

$$\dot{\mu} = \rho\mu - \mu G'(x). \tag{4}$$

Transversality conditions must also be checked, which is done below.

Control condition (3) may be rewritten as:

$$h = \max\left(0, c^{-1}(p-\mu)\right), \tag{5}$$

and implies that with $p < \mu \Rightarrow h = 0$. Inserting from Eq. (5) into the growth equation yields next:

$$\dot{x} = G(x) - \max\left(0, c^{-1}(p-\mu)\right). \tag{6}$$

We can use Eqs. (4) and (6) to obtain a phase diagram in the (x, μ)-space. The isocline for $\dot{x} = 0$ may be constructed as follows:

$$\begin{gathered} \dot{x} = G(x) - \max\left(0, c^{-1}(p-\mu)\right) = 0 \\ \Updownarrow \\ \mu = \varphi(x) = p - c\big(G(x)\big) \;\forall \{x : h > 0\} \end{gathered} \tag{7}$$

Note that $\varphi(0) = \varphi(K) = p$ and that $\varphi(x) < p$ for all $x \in (0, K)$. The isocline for $\dot{\mu} = 0$ is given by:

$$\begin{gathered} \dot{\mu} = \rho\mu - \mu G'(x) = 0 \\ \Updownarrow \\ \mu = 0 \text{ or } x = G'^{-1}(\rho) \end{gathered} \tag{8}$$

We shall assume that there is a pair (x_{ss}, μ_{ss}) that solves the equations $\dot{x} = 0$ and $\dot{\mu} = 0$ and hence define the equilibrium (steady state) of our model. The isoclines in (7) and (8) are depicted and discussed in Figure 1 where we illustrate two possible paths for a stable manifold when x is lower than its steady state value. This entails one path where the stable manifold lies below the line $\mu = p$ except at $(x, \mu) = (0, p)$ where they intersect and a second possible manifold which crosses the line $\mu = p$ at some stock value in the interval $(0, x_{ss})$. There can only be one stable manifold, so we have to choose between them. This is done in Proposition 1.

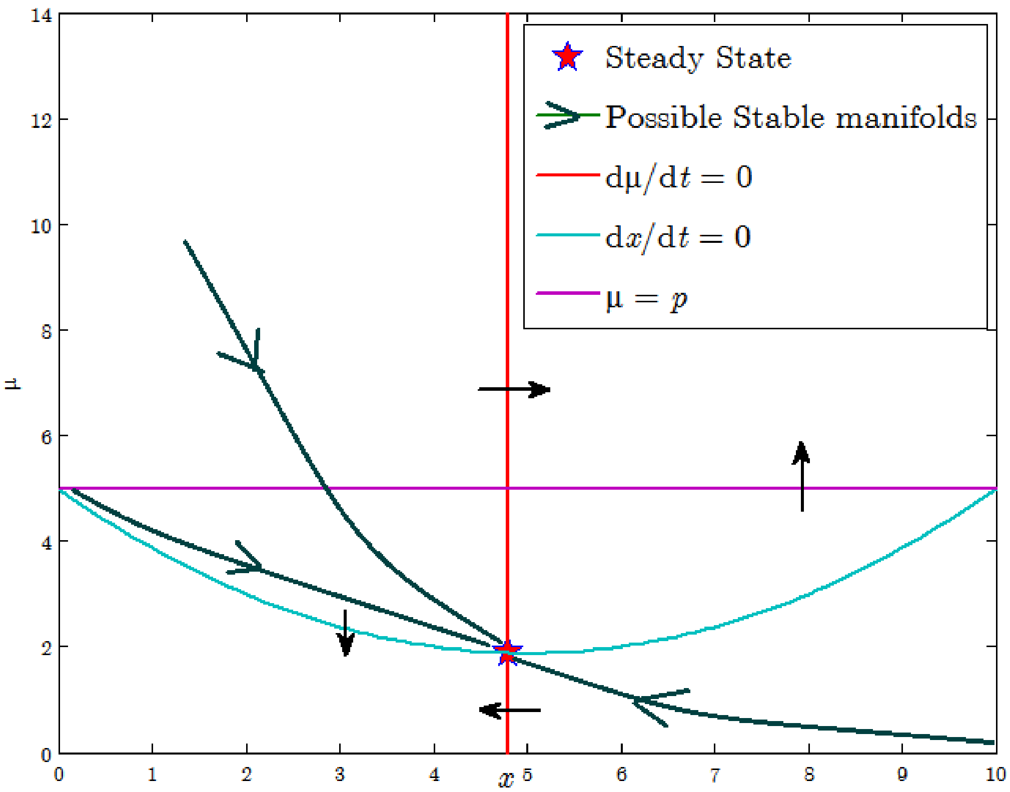

Figure 1. Isoclines in a phase diagram in the (x, μ)-space. The isoclines at x = 0, x = K and μ = 0 are not drawn. The black arrow indicates system directions of the isoclines. The star indicates the steady state point (x_{ss}, μ_{ss}) that solves the equations dx/dt = 0 and dμ/dt and it follows from the directions of the black arrows crossing the isoclines that it is a saddle point as expected. Lines with arrows indicating directions are possible stable manifolds. For values of x below the steady state there are two possibilities. One where the stable manifold lies starts above the μ = p line, and one where the entire line lies below the μ = p line. If this last possibility is the case, the stable manifold must start at the point (x, μ) = (0, p). Proposition 1 shows that this is impossible, so the stable manifold must start at some point where μ > p.

Proposition 1.
Any path originating from (x, μ) = (0, p) can not be a stable manifold.

Proof: The slope of the stable manifold at (x, μ) = (0, p) is given by:

$$\begin{aligned}(d\mu/dx)_{(x,\mu)=(0,p)} &= \dot{\mu}/\dot{x} = \frac{p(\rho - G'(0))}{G(0) - \max\left(0, c^{-1}(p-p)\right)} \\ &= \frac{p(\rho - G'(0))}{G(0)} = -\infty\end{aligned} \tag{9}$$

This holds under our assumption of an intrinsic growth rate $G'(0)$ that exceeds the rate of discount, and $G(0)=0$. This slope is clearly smaller than the finite slope of the isocline for $\dot{x}=0$, so a stable manifold would enter into the area below the isocline for $\dot{x}=0$, which implies that the stable manifold cannot go through the steady state.

■

Proposition 1 has a powerful implication that we sum up in a proposition although the proof is very simple.

Proposition 2.
There exists a non-empty interval [0, x^*] where it is optimal to set $h = 0$.

Proof: It follows from Proposition 1 that there exists a stock level x^* where the downward sloping stable manifold crosses the line $\mu = p$, and therefore $h = 0$ for all $x \in [0, x^*]$.

In Nævdal (2016) it was proven that if revenue is linear in harvest, the shadow price would go to infinity as the stock approaches zero. The proof of this result hinged on the harvest rate being zero if stocks are below the steady state level. Proposition 1 above implies that the proof in Nævdal (2016) may be generalized to the case where harvest costs are strictly convex and also in this more general case, the shadow price will go to infinity as the stock approaches zero. This is done in Proposition 3. Proposition 3 thinks of the stable manifold in a slightly unusual manner. The stable manifold is a continuous mapping from x to μ and it thus makes sense to think of μ as a function of x. We can then use the ratio $\dot{\mu}/\dot{x}$ and steady state conditions to construct a differential equation with boundary conditions (Judd 1998, Ch.10.7).

Proposition 3.
Along the stable manifold $\lim_{x\downarrow 0} \mu(0) = \infty$ holds.

Proof: Let (x_{ss}, μ_{ss}) be the known steady state level of the optimally managed system defined by problem (1). Over the interval [x^*, x_{ss}] one can find the stable manifold by solving the differential equation:

$$\frac{\dot{\mu}}{\dot{x}} = \frac{d\mu}{dx} = \frac{\rho\mu - \mu G'(x)}{G(x) - c^{-1}(p-\mu)},$$

with the boundary condition $\mu_{ss} = \mu(x_{ss})$. By Proposition 2 there exists an x^* such that $\mu(x^*) = p$. One can therefore find the solution for $\mu(x)$ over the interval [0, x^*] by solving the following differential equation:

$$\frac{\dot{\mu}}{\dot{x}} = \frac{d\mu}{dx} = \frac{\rho\mu - \mu G'(x)}{G(x)}, \ \mu(x^*) = p.$$

Nævdal (2016) showed that this equation has the solution:

$$\mu(x) = \frac{pG(x^*)}{G(x)} \exp\left(-\int_x^{x^*} \frac{\rho}{G(\eta)} d\eta\right) \tag{10}$$

and that $\lim_{x\downarrow 0} \mu(x) = \infty$. The proof is reproduced in the in the Appendix.

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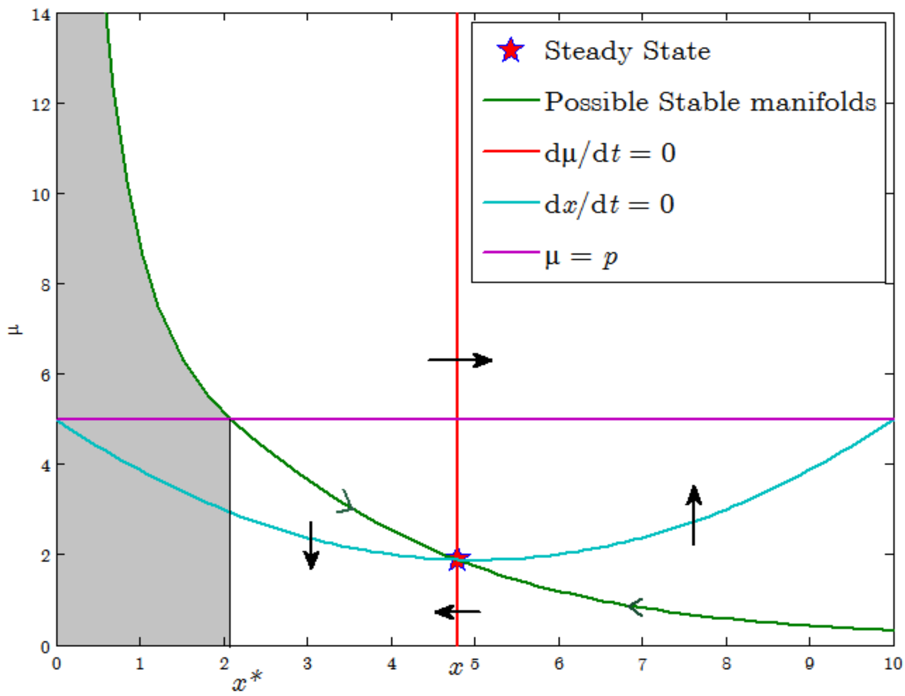

Figure 2. Computer generated phase diagram for the model in Equation (1). Note that μ along the stable manifold increases as *x* goes to zero. It does in fact go to infinity. It crosses the line $h(x,\mu) = 0$ at x^*. From (5) it should be clear that $h(x,\mu) = 0$ implies that $\mu = p$ at x^*. Thus at all $x \leq x^*$ it holds that $h = 0$. The stable manifold is in fact the derivative of the value function. As $V(0) = 0$, the area under the stable manifold is therefore the value function. The shaded area shows $V(x^*)$, which is the value of the fishery at the stock level x^*. Although Propositions 1-3 hold for general growth and cost functions with the properties stated above, the graph is drawn using a logistic growth function $G(x)=rx(1-x/K)$ with $K = 10$ and $r = 1$, and the current profit function $ph - C(h) = 5h - ½h^2$. The discount rate has been set to $\rho = 0.05$.

Propositions 1, 2 and 3 enables us to draw the more complete phase diagram, depicted in Figure 2. It is worthwhile to note that as the stable manifold entails the allowable combinations of *x* and μ along an optimal path, it may in fact be interpreted as a function $\mu(x)$ that gives the derivative of the value function, $\mu(x) = \partial V(x)/\partial x$. As the value function $V(x)$ clearly must satisfy $V(0) = 0$, the value function can be demonstrated in the phase diagram as the area below the stable manifold as indicated by the shaded area in Figure 2.

Downward sloping demand curve

Above it was assumed that the unit fish price was fixed. If we introduce a downward sloping linear demand curve, we find that this yields very much the similar solution as the situation when the price is fixed. Indeed, in a model with costless fishing and a linear demand curve $p(h) = a - bh$, with the choke price $a > 0$ and $b > 0$, the analysis will be identical, but where the choke price replaces *p*.

With costs included, the current profit writes $[p(h)h - C(h)]$. Under monopolistic exploitation, the control condition (3) is now replaced by:

$$\partial H/\partial h = p'(h)h + p(h) - c(h) - \mu \leq 0 \ (= 0 \ if \ \ h > 0)$$

This implies $p(0) - \mu < 0 \Rightarrow \ h = 0$ which also has identical structure to the model analyzed in section 2. The same will be the case when we imagine a social planner maximizing the present-value sum of consumer surplus and producer profit. The control condition reads then $\partial H/\partial h = p(h) - c(h) - \mu \leq 0 \ (= 0 \ if \ \ h > 0)$ which also implies $p(0) - \mu < 0 \Rightarrow \ h = 0$.

Positive and finite marginal cost at zero harvest, $c(0) > 0$.

In the baseline model analysed in section 2, the proof was based on there being a single point at $x = 0$ where the stable manifold could cross the μ-axis if the stable manifold did not lie above the line $\mu = p$. If instead $c(0) > 0$ and finite, there is in fact an interval on the μ-axis where the stable manifold could start. This interval is given by $(p - c(G(x), p)$. If the stable manifold actually does origin from this interval, it would imply that an increase in marginal cost would increase harvest levels for low stock values, something that must be considered a contradiction. A definite proof is given in Proposition 4.

Proposition 4. No stable manifold can start in the open interval $(p - c(G(x), p)$ on the μ-axis.

Proof: If the stable manifold starts in the open interval $(p - c(G(x), p)$, then $h = 0$. But then, using techniques used in Proposition 3, one can show that $\lim_{x \downarrow 0} \mu(0) = \infty$ which is a contradiction.

3. Age structured models

We now examine age structured models in order to see if the results from above carry over. In particular, we want to check whether the shadow price goes to infinity as the stock approaches zero, and whether this also implies that no harvesting will occur at low stock levels.

Recent years have seen increased interest in age structured models and the implications of dropping lumped parameter models (see, e.g., Tahvonen 2009, and Skonhoft et al. 2012). Typically, the cohort length of a fish stock is measured in years as the relevant time scale piscine reproduction usually occurs one a annual basis. This would require a biological model with several cohorts with year-class specific contribution to recruitment and year-class specific natural survival as well as harvest rates. Indeed, cohort models quickly become analytically intractable. Here we first analyse two simplified cases. One case where the adult period is relatively short compared to the time span of a young animal. This may correspond to e.g. salmon where most of the species' life history is in the native river (2-4 years) before migrating into the ocean and spending 1-3 year there before returning back to spawn in its native river. After spawning it dies (about 90 %). It is only at the end of this last period that salmon are harvested in significant numbers. The second case we analyse is where the period as a young is short relative to the time (potentially) spent as an adult. This may correspond to e.g. pelagic cod which only becomes old enough to be harvested and spawn at the age of 3 years and may live to become more than 20 years old. In these two particular cases we can use the differences in the time span of cohorts to utilise slow/fast-dynamics in order to simplify the analysis, Crépin (2007), Guttormsen et al (2008). Based on these examples we examine a very simple age structured model with two cohorts, young, x, and adult, y. It is assumed that only adult fish are harvested. The equation of motion for young fish is assumed given by:

$$\dot{x} = ry\left(1-\frac{x}{K}\right)-\delta x \tag{11}$$

Larvae production is proportional to the stock of adults, but survival is density dependent. A fraction δ enters the stock of adults every unit of time. The stock of adults grow according to

$$\dot{y} = \delta x - \gamma y - h \tag{12}$$

Here γ is the natural mortality rate and h is harvesting. In the absence of harvesting one can show that the steady state is given by:

$$x = \frac{K(r-\gamma)}{r} \quad \text{and} \quad y = \frac{\delta K(r-\gamma)}{\gamma r}. \tag{13}$$

Positive steady state levels without harvesting require $r > \gamma$ which is assumed. Also, in order to simplify the analysis it is assumed that instantaneous profit from harvesting is given by $ph - ½kh^2$ where the fish price p and the effort cost parameter k is now are related to effort levels.

Salmon

Since salmon is characterised by a long period as young before experiencing a short period as adult and dying after spawning, we can treat x as a slow variable and y as fast variable. The implication of y being a fast variable is that y moves very quickly from one steady state to another relative to x. The most straightforward way to model this is to let movement from one steady state to another be instantaneous implying that $\dot{y} = 0$ or $y = (1/\gamma)(\delta x - h)$ from Eq. (16). Inserting into Eq. (11) gives then:

$$\dot{x} = (r/\gamma)(\delta x - h)\left(1-\frac{x}{K}\right)-\delta x,\ x(0) \text{ given.} \tag{14}$$

With our profit function, the management problem is then to maximise (1) subject to (14). The Hamiltonian for this problem is:

$$H = ph - \frac{k}{2}h^2 + \mu\left((r/\gamma)(\delta x - h)\left(1-\frac{x}{K}\right)-\delta x\right) \tag{15}$$

Necessary conditions for optimality include:

$$h(x,\mu) = \max\left[0, \frac{p}{k} - \frac{r(K-x)\mu}{k\gamma K}\right] \tag{16}$$

and

$$\dot{\mu} = \rho\mu - \frac{\left(h(x,\mu)r + \left(K(r-\gamma)-2rx\right)\delta\right)}{K\gamma}\mu. \tag{17}$$

Note that the expression for $\dot{\mu}$ depends on h. Thus when calculating $\dot{\mu}$ one must be careful to insert $h = 0$ whenever prescribed by μ and x. The boundary between values of x and μ where $h > 0$ and $h = 0$ is given by:

$$h \geq 0 \Leftrightarrow \mu \leq \frac{p\gamma K}{r(K-x)} \tag{18}$$

Given our assumptions, there are three possible steady states. If r/γ is sufficiently high, there is a steady state with $\mu = 0$, $h = p/k$. This only applies when the growth of the fish stock is so robust that capital theoretic considerations do not apply and will not be analysed further. These steady states are given by:

$$h_{low} = \frac{K\left(r\delta + \gamma\delta - \sqrt{\gamma}\sqrt{4r\delta^2 + \gamma\rho^2}\right)}{r}, \; x_{low} = \frac{K\left(2r\delta + \gamma\rho - \sqrt{\gamma}\sqrt{4r\delta^2 + \gamma\rho^2}\right)}{2r\delta} \tag{19}$$

$$h_{high} = \frac{K\left(r\delta + \gamma\delta + \sqrt{\gamma}\sqrt{4r\delta^2 + \gamma\rho^2}\right)}{r}, \; x_{high} = \frac{K\left(2r\delta + \gamma\rho + \sqrt{\gamma}\sqrt{4r\delta^2 + \gamma\rho^2}\right)}{2r\delta} \tag{20}$$

Steady state expressions for μ may also be computed, but are too complicated to be informative. It should be clear from (19) and (20) that $x_{low} < x_{high}$. Further, in order for steady states to be positive we must have that

$$\rho < \delta(r - \gamma)$$

We can use these conditions to draw a phase diagram. This is done in Figure 3 and discussed in the caption.

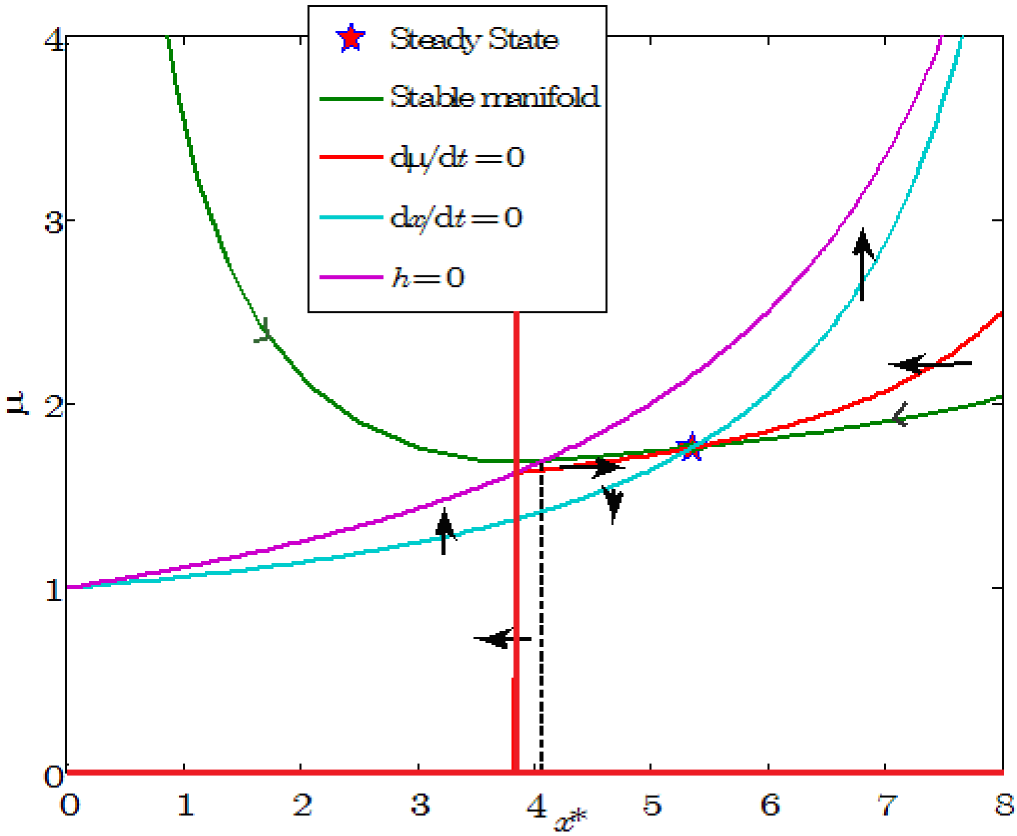

Figure 3, Phase diagram for cohort fishery with fast slow dynamics. At $x = x^*$, the stable manifold crosses the $h = 0$ boundary. Thus if $x < x^*$ it is optimal to set harvest levels to zero. This implies that Proposition 3 applies also in this case and $\lim_{x \downarrow 0} = \infty$.

Cod

Here it is assumed that it is y that is the slow variable and x that moves instantaneously from steady state to steady state. Therefore we set $\dot{x} = 0$ and obtain:

$$x = \frac{Kry}{ry + K\delta} \tag{21}$$

Inserting Eq, (24) into the expression for $\dot{y}$ implies that Eq. (12) may be written as:

$$\dot{y} = \frac{Kry\delta}{ry + K\delta} - \gamma y - h\,. \tag{22}$$

We may then formulate the current value Hamiltonian:

$$H = ph - \frac{k}{2}h^2 + \mu\left(\frac{dKry}{dK + ry} - \gamma y - h\right). \tag{23}$$

Using standard techniques we may then calculate the steady state that follows from using optimal policy:

$$h_{SS} = \frac{\delta Kr(r+\gamma)(\gamma+\rho) - (2\gamma+\rho)\sqrt{\delta^2 K^2 r^3(\gamma+\rho)}}{r^2(\gamma+\rho)},\; y_{SS} = \frac{\sqrt{\delta^2 K^2 r^3(\gamma+\rho)} - \delta Kr(\gamma+\rho)}{r^2(\gamma+\rho)} \tag{24}$$

It is straightforward to show that these expressions are positive as long as $r - \gamma > \rho$. One can also here draw a phase diagram, but it turns out to be almost identical to Figure 2 and is therefore omitted. Therefore, the main conclusions from Figure 2 carry over, namely that there exists a x^* such that h is zero for $x \leq x^*$ and that the shadow price also in this case will go to infinity as x approaches zero.

Both our cohort models are variations over the same underlying cohort structure. The results are the same. The shadow price goes to infinity for low stock levels and therefore there is some lower bound on stock levels where harvest rates are zero. Thus we confirm analytically numerical results from e.g. Tahvonen (2008) and Skonhoft et al (2012) who find zero harvesting at low levels.

Numerical analysis when dynamics is on the same time scale.

In order to demonstrate that zero harvest for low stocks is not an artefact of choosing slow/fast-dynamics we also present a numerical analysis where the model with population dynamics given by (11) and (12) are assumed to be valid on the same time scale. This is done in Figure 4 and discussed in the caption.

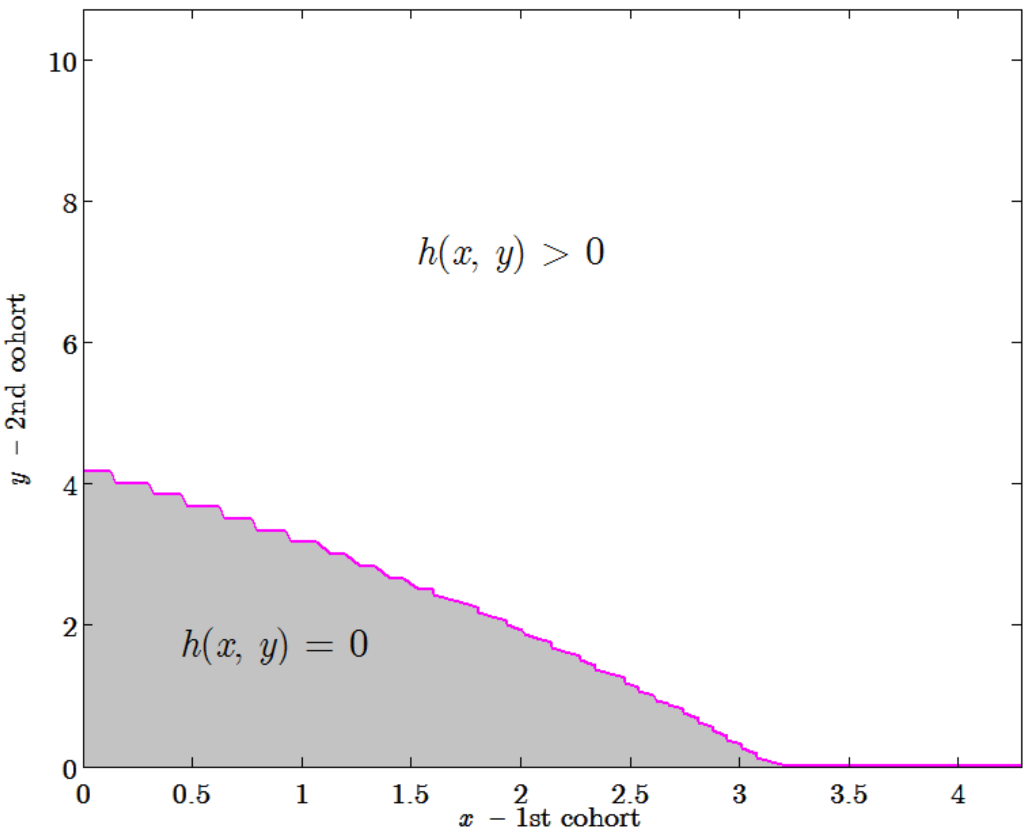

Figure 4. The solution to (1) when population dynamics are given by (11) and (12) can be found through dynamic programming as a feedback control where optimal harvest rate can be written as a function of *x* and *y*. The figure shows those values of *x* and *y* where optimal harvest is zero. For low values of *x* and *y* the shadow price of *y* is sufficiently high that it is optimal harvest is zero. The plot is generated using $C(h) = \frac{1}{2}ch^2$, $p = 10$, $c = 1$, $r = 1$, $K = 10$, $\rho = 0.05$, $\delta = 0.5$ and $\gamma = 0.2$.

4. Concluding remarks

In this paper we have examined the basic nonlinear control variable fishery model originating from Clark and Munro (1975), and demonstrated under what circumstances it is optimal to stop harvesting when the stock becomes sufficiently low. The main assumptions in our modelling are that the intrinsic (maximum) growth rate of the fish stock exceeds that of the discount rent, that the growth of the fish stock is zero when the fish stock is zero and that the marginal net benefit is finite for all harvest levels, and particularly for zero harvest

The paper provides four Propositions and these enable us to draw a more complete phase diagram than what is found in, among others, Clark (1990) and Leonard and Long (1992). The most important of these propositions from a management perspective, is that it always exists a strictly positive stock level below which it is optimal with no harvest. This is perhaps not too surprising. If the value of fish stock grows faster in the ocean than it does in the bank, we would prefer to have the fish staying in the ocean until it has grown to the point where the return in the ocean is equal to returns in the bank. The non-negativity constraint on harvesting implies that we cannot put fish into the lake. However, the fact that optimal harvest levels is always zero for low stock levels also imply that the shadow price of the stock will always go to infinity as the stock goes to zero. This was proved for a simple model with harvesting cost depending on the harvest only, but showed through numerical examples that it also applies to age structured fisheries.

It is also demonstrated how the value function may be illustrated in the phase diagram. Our basic model is formulated for a schooling fishery. Stock dependent costs as well as some other extensions are also included in the analysis.

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Appendix

The solution to the differential equation in Proposition 3..

Dividing the differential equation by μ and integrating over $[x, x^*)$ gives:

$$\int_x^{x^*} \frac{1}{\mu(x)} \frac{d\mu}{d\eta} dy = \int_x^{x^*} \frac{\rho}{G(\eta)} d\eta - \int_x^{x^*} \frac{G'(\eta)}{G(\eta)} d\eta$$

$$-\int_{\mu(x^*)}^{\mu(x)} \frac{1}{\mu} d\mu = \int_x^{x^*} \frac{\rho}{G(\eta)} d\eta - \left(\ln\left(G(x^*)\right) - \ln\left(G(x)\right)\right)$$

$$\ln\left(\frac{\mu(x)}{\mu(x^*)}\right) = -\int_x^{x^*} \frac{\rho}{G(\eta)} d\eta + \ln\left(\frac{G(x^*)}{G(x)}\right)$$

$$\frac{\mu(x)}{\mu(x^*)} = \frac{G(x^*)}{G(x)} \exp\left(-\int_x^{x^*} \frac{\rho}{G(\eta)} d\eta\right)$$

Inserting for $\mu(x^*) = p$ and rearranging gives the expression for $\mu(x)$.

$$\mu(x) = \frac{pG(x^*)}{G(x)} \exp\left(-\int_x^{x^*} \frac{\rho}{G(y)} dy\right)$$

This solution is only valid over the interval (0, x*]. Note that the integral in this expression converges, $\mu(0)$ is clearly infinite. If the integral does not converge, the expression is on the form “0/0” and must be evaluated with L'Hôpital's rule.

Calculating μ(0)

Applying L'Hôpital's rule yields

$$\lim_{x\to 0} \mu(x) = pG(x^*) \frac{\lim_{x\to 0} \frac{d}{dx}\left(\exp\left(-\int_x^{x^*} \frac{\rho}{G(\eta)} d\eta\right)\right)}{\lim_{x\to 0} G'(x)}$$

$$= pG(x^*) \frac{\lim_{x\to 0} \exp\left(-\int_x^{x^*} \frac{\rho}{G(\eta)} d\eta\right) \frac{\rho}{G(x)}}{\lim_{x\to 0} G'(x)}$$

$$= pG(x^*) \lim_{x\to 0} \frac{\rho}{G'(x)} \times \lim_{x\to 0} \frac{\exp\left(-\int_x^{x^*} \frac{\rho}{G(\eta)} d\eta\right)}{G(x)}$$

$$= \frac{\rho}{G'(0)} \lim_{x\to 0} \frac{pG(x^*)}{G(x)} \exp\left(-\int_x^{x^*} \frac{\rho}{G(\eta)} d\eta\right)$$

The last line implies that:

$$\lim_{x\to 0} \mu(x) = \frac{\rho}{G'(0)} \lim_{x\to 0} \mu(x)$$

This can only be true if $\mu(0) = 0$ or $\mu(0) = \infty$. But because $\dot{\mu} < 0$ in a neighbourhood around $x = 0$ and $G(x) > 0$ it must be true that for x close to zero $\mu'(x) = \dot{\mu}/\dot{x} < 0$, which implies that $\mu(0) = \infty$.

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