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Catastrophes and *ex post* shadow prices—How the value of the last fish in a lake is infinity and why we should not care (much)☆

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ABSTRACT

Catastrophic risk is currently a hotly debated topic. This paper contributes to this debate by showing two results. First, it is shown that for a certain class of optimal control problems, the derivative of the value function with respect to the initial state may approach infinity as the state variable goes to zero, even when the first-order partial derivatives of the integrand and transition functions are finite. In the process, it is shown that standard phase diagrams used in optimal control theory contain more information than generally recognized and that the value function itself may be easily illustrated in these diagrams. Second, we show that even if the value function has an infinite derivative at some point, it is not correct to avoid this point in finite time at almost any cost. The results are illustrated in a simple linear-quadratic fisheries model and proven for a more general class of growth functions.

1. Introduction

The Economic Management of Catastrophic Risk (EMCR) is currently hotly debated, and several issues have not been resolved. Catastrophic risk is usually discussed in a dynamic framework in which the catastrophe is some detrimental event whose probability of occurrence is distributed over time. EMCR often applies dynamic optimization techniques in order to derive management rules. The standard approach to solving such problems is to first calculate the *ex post* solution, which is a contingency plan for optimal management after a catastrophe has occurred. Using the *ex post* solution as a building block, one then solves the *ex ante* management problem, which is the optimal program to be followed as long as the catastrophe does not occur (Clarke and Reed, 1994; Tsur and Zemel, 1995; Tsur and Zemel, 1998).

The present paper aims to elaborate on how the marginal value of a resource *after* a catastrophe, termed *ex post* shadow price, affects optimal management *before* the catastrophe. Two important results are derived that are the main theoretical contributions of this paper. First, it is shown that *ex post* shadow prices can become infinite even if all of the first-order

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partial derivatives of the integrand and transition functions have finite values on their domains. Second, it is shown that *ex post* shadow prices have limited or no impact on *ex ante* regulation if a catastrophe is defined as a total collapse in a resource stock.

The importance of *ex post* shadow prices for optimal management is analyzed in a number of papers. For example, de Zeeuw and Zemel (2012) analyze a pollution control problem in a dynamic programming framework and find that *ex post* shadow prices are important for *ex ante* regulation. Polasky et al. (2011) find the same in a fisheries model. As shown below, the results in these papers are caused by the authors not specifying the catastrophe as severe enough. When a catastrophe is defined as a total collapse in a resource, the *ex post* shadow price will, in general, be of no or relatively minor importance for optimal management before the catastrophe.

There seems to be a certain amount of confusion about how *ex post* shadow prices are generated and affect management. A prominent example is Weitzman's (2009) contribution. This paper has been much debated since its publication. The "dismal theorem" presented therein states that under certain conditions, the process of gathering data about a stochastic process leads to a situation where expected marginal utility explodes. Weitzman is careful to state that his result does not depend on "a mathematically illegitimate use of the symbol +∞" but arises naturally and that criticism of his results by "somehow discrediting this application of expected utility on the narrow grounds that infinites are not allowed in a legitimate theory of choice under uncertainty" is unfounded. This paper will show that infinite marginal values are most certainly allowed in such a theory, as they, under certain conditions, turn up endogenously. Thus, the data gathering process applied by Weitzman (2009) is not required to generate infinite shadow prices, even when all functional forms have everywhere finite derivatives. The realism of Weitzman's results have been challenged by, for example, Nordhaus (2011), Pindyck (2011) and Costello et al. (2010).

The question is then if infinite shadow prices or infinite marginal utility actually matter. Weitzman himself argues that "The burden of proof if in climate change is presumably upon whoever calculates expected discounted utilities without considering that structural uncertainty might matter more than discounting or pure risk. Such middle-of-the-distribution modeler should be prepared to explain why the bad fat tail of the posterior predictive PDF does not play a significant role in climate-change CBA when it is combined with a specification that assigns high disutility to high temperatures." This statement must be qualified. The dismal theorem refers to expected *marginal* utilities and not the *level* of disutility. However, even so, the dismal theorem seems to indicate that getting into a situation where marginal utility is infinity should be avoided at almost any cost. Indeed, when Inada conditions specify that functions evaluated at zero have infinite derivatives, the purpose is to ensure that a model economy converges to an interior steady state. One important implication of the present paper is that Inada conditions are not required to guarantee such an outcome as infinite shadow prices turn up endogenously. This property to some extent carries over to stochastic models, as it has been known since the work of Brock and Mirman (1972) that optimal paths converge to a uniquely non-trivial stationary solution and that this result depends on imposing Inada-conditions. Kamihigashi (2006) shows that if Inada conditions are not imposed, there will almost surely be convergence to zero. One could therefore be excused for thinking that imposing Inada conditions guarantee against it ever being optimal to reach a state with infinite shadow price, typically a state where the amount of some valuable variable is equal to zero. Mirman and Zilcha (1976) provide an example where consumption is not bounded away from zero even if Inada-conditions are imposed. It is shown in the following that this turns out to be wrong in models where a catastrophe is defined as a total collapse in the state variable. The Inada conditions are sufficient to eliminate the possibility of reaching a state with infinite shadow prices through an *incremental* reversible process, but such results do not apply to the type of major shocks we associate with large rapid disruptions such as the total collapse of a fish stock or the end of a civilization through a total collapse of capital stocks.

2. How infinite shadow prices can turn up endogenously—why the last fish in the lake is worth infinity

Consider a general autonomous optimal control problem with infinite time horizon where the instantaneous utility is $F(u, x)$; the stock grows according to $\dot{x} = f(x, u)$, where $A = [0, a) \subseteq [0, \infty)$ is the feasible set of x values and initial conditions $x(0)$ are given.

Thus, we are examining the following problem:

$$\max_{u \in U \subseteq \mathbb{R}} \int_0^{+\infty} F(x, u) e^{-\rho t} \mathrm{d}t \qquad (1)$$
$$\text{s.t} \dot{x} = f(x, u), x(0) = x_0$$

Optimality conditions for problems like Eq. (1) may be found in Theorem 9.11.1 in the work of Sydsæter et al. (2005), where the Hamiltonian is assumed to be non-convex in x and u. The value of u that maximizes the Hamiltonian can be written $u(x, \mu)$, where μ is the co-state variable or shadow price on x. Optimality requires that optimal time paths for x and μ satisfy:

$$\dot{\mu} = \rho\mu - F'_x(x, u(x, \mu)) - \mu f'_x(u(x, \mu), x) \qquad (2)$$

and

$$\dot{x} = f(x, u(x, \mu)) \tag{3}$$

Additionally, a transversality condition must hold. Here, we shall assume the existence of a steady state value of x, denoted x_{ss} in A and a corresponding finite steady state shadow price μ_{ss}. This assumption implies that the transversality condition is automatically satisfied, something that must be checked in specific applications.

The conditions in Eqs. (2) and (3) can be solved for optimal time paths, but rather than explicitly calculate these time paths for x and μ, one may reduce the dimensionality of the problem and solve a differential equation where the unknown function is $\mu(x)$. A conceptually straightforward method for doing this is to solve the following differential equation:

$$\frac{\dot{\mu}}{\dot{x}} = \frac{d\mu}{dx} = \frac{\rho\mu - F'_x(x, u(x, \mu)) - \mu f'_x(x, u(x, \mu))}{f(x, u(x, \mu))},\ \mu(x_{ss}) = \mu_{ss} \tag{4}$$

What Eq. (4) achieves is the elimination of time as a variable and the reduction of the problem to solving a differential equation where the unknown function is the shadow price as a function of the state variable along an optimal path. As we shall see below, the resulting function $\mu(x)$ is the stable manifold for a phase diagram in (x, μ)-space and also the derivative of the value function.

Actually computing Eq. (4) can be tricky. Analytical solutions can only be found for a few special cases, and numerical solutions must recognize that evaluating the differential equation at $\mu(x_{ss}) = \mu_{ss}$ implies evaluating a "0/0" expression. See Judd (1998), chapter 10.7, for workarounds. The expression in Eq. (4) will be quite important for what follows.

I claim that there are three cases that may lead to $\lim_{x\to 0}\mu(x) = \infty$ because they all imply that $d\mu/dx_{x\to 0} = \infty$:

1. $F'_u(0,0) = \infty$. This case is well known and is the case that is used in the dismal theorem debate.
2. The second case is when the numerator in Eq. (4) is infinite when evaluated at $x = 0$. This case is discussed in the supplementary material.
3. The last possibility is when the denominator in Eq. (4) goes to zero as x goes to zero. That is, when $f(0, u(0, \mu)) = 0$.

Note that $d\mu/dx_{x\to 0} = \infty$ is in itself not a proof for $\lim_{x\to 0}\mu(x) = \infty$. Cases 1 and 2 require that the model uses specific functional forms with infinite derivatives. Case 3, which to the best of my knowledge has not been discussed in the literature, requires no such thing and would occur quite naturally in a number of settings. Most models of renewable resources have an assumption built in that growth in the resource is zero when the stock is zero. Aggregate macro-production functions assume that capital is required to create more capital. The main point here is that there are a number of naturally occurring processes that have a self-generative nature and whenever one performs dynamic economic analysis of them, one must accept that there is a possibility that the current value shadow price of the resource goes to infinity as its stock goes to zero. This is demonstrated in the sequel for a fisheries model. An example from macroeconomics is given in the supplementary material.

To show that Case 3 may generate infinite shadow prices, we examine a version of a well-known fisheries model. Clark and Munro (1975) present a model that gives the theory of renewable resources a proper capital theoretic foundation and serves as a cornerstone of resource economics. The simplest version of the textbook dynamic fishery model assumes an exogenous price of fish, $p > 0$; thus, $F(x, u) = pu$. Harvesting, $u \in [0, \bar{u}]$, is assumed to be costless and the equation of motion is given by:

$$\dot{x} = f(x, u) = G(x) - u \tag{5}$$

It is assumed that $G(0) = 0$ and that there exists a positive constant K such that $G(K) = 0$. Further, $G(x)$ is assumed to be differentiable over $[0, K]$ and strictly positive over $(0, K)$, where K is the carrying capacity. This leads to the optimization problem

$$V(x(0)) = \max_{h(t)} \int_0^\infty pue^{-\rho t}\mathrm{d}t \tag{6}$$

subject to Eq. (5), $x \geq 0$ and $x(0)$ given. It is assumed that $\rho < G'(0) < \infty$. This model is well known and easy to analyze with optimality conditions such as the sufficiency conditions found in Theorem 9.11.1 in the work of Sydsæter et al. (2005). The Hamiltonian is given by:

$$H = pu + \mu(G(x) - u) \tag{7}$$

Here, μ is the co-state variable. This leads to the following optimality conditions:

$$\begin{aligned} &p > \mu \leftrightarrow u = \bar{u} \\ &p = \mu \leftrightarrow u = G(x) \\ &p < \mu \leftrightarrow u = 0 \end{aligned} \tag{8}$$

$$\dot{\mu} = \rho\mu - \mu G'(x) \tag{9}$$

Strictly speaking, when $\mu = p$, one is indifferent between different values of u. We here follow the common practice to choose the particular u that sets $\dot{x} = 0$ so that no further change in u is required. This is equivalent to restricting the set of feasible controls to piecewise deterministic functions.

Combining Eqs. (8) and (9) with the appropriate transversality condition determines the optimal program. Following the procedure from Eq. (4) one can combine Eqs. (7)–(9) and the steady state solution, which in this particular case is $\mu(x_{ss}) = p$, and construct a single differential equation for μ as a function of x. This is done in Proposition 1.

Proposition 1. Assume that $\bar{u}$ is sufficiently large so that for all x there exists a $u \in [0, \bar{u}]$ that solves $G(x) = u$. Then, there exists a steady state given by $(x_{ss}, \mu_{ss}) = (x^*, p) \in \mathbb{R}^2_{++}$ and $\mu(x)$ is given by:

$$\mu(x) = p\frac{G(x*)}{G(x)} \exp\left(-\int_x^{x*} \frac{\rho}{G(y)}dy\right) \tag{10}$$

Proof. By assumption, there exists a value x^* such that $\rho = G'(x*)$ for some $x^* \in [0, K]$. Steady state requires that $G(x^*) - u = 0$, which exists if $\bar{u}$ is large enough to satisfy the premise of the proposition. From Eq. (8), it is evident that this requires that $\mu = p$ in steady state. From Hartl (1987), it follows that a path toward the steady state will be monotonic, which given Eq. (8) can only happen if $u = 0$ for all $x < x^*$. Thus for all $x \leq x^*$, Eq. (4) may be written:

$$\frac{\dot{\mu}}{\dot{x}} = \frac{d\mu}{dx} = \frac{\rho\mu - \mu G'(x)}{G(x)}, \ \mu(x*) = p \tag{11}$$

This is a separable differential equation, and the solution is shown in the Appendix A to be given by Eq. (10). □

Having found an expression for the shadow price, we can prove that in standard economic parlance, the last fish in the lake is indeed worth infinity.

Proposition 2. Assume that $\rho < G'(0)$. Then, $\lim_{x\to 0}\mu(x) = \infty$.

Proof. From Eq. (10), $\mu(x)$ is clearly non-negative for all $x \in (0, x^*)$ because it is a product of terms that are all non-negative. There are two cases to consider. If the integral in Eq. (10) converges, then the proof is trivial as $G(0)$ is zero. If the integral does not converge, we have a "0/0" expression, which can be evaluated using L'Hôpital's rule. It is shown in the Appendix A that application of L'Hôpital's rule yields the following expression:

$$\lim_{x\to 0}\mu(x) = \frac{\rho}{G'(0)}\lim_{x\to 0}\mu(x) \tag{12}$$

Eq. (12) implies that $\mu(0)$ is either zero or infinity. As $\rho < G'(0)$, we know that $\dot{\mu} < 0$ for values of x in the interval $[0, \varepsilon]$, and $G(x) > 0$ in the same interval. Thus $\mu'(x) = \dot{\mu}/\dot{x}$ must be negative over the same interval. This is only possible if $\lim_{x\to 0}\mu(x) = \infty$. □

This result is quite strong. In a process where instantaneous utility is linear in the control, the shadow price goes to infinity regardless of the shape of the growth function as long as $G(0) = 0$.

Having established $\lim_{x\to 0}\mu(x) = \infty$ in a simple model, we can easily extend the result to more general models. For example, if instantaneous utility can be written, *e.g.*, as $pu + H(x)$ where $0 < H'(x) < \infty$, then Eq. (11) becomes:

$$\frac{\dot{\mu}}{\dot{x}} = \frac{d\mu}{dx} = \frac{\rho\mu - H'(x) - \mu G'(x)}{G(x)} \tag{13}$$

This expression is steeper than Eq. (11) as x goes to zero, which implies that also in this case, $\lim_{x\to 0}\mu(x) = \infty$. A more general proof can be found by straightforward inspection of the Hamilton–Jacobi–Bellman equation, but it requires that the value function be differentiable.

The result in Proposition 2 may be illustrated in a phase diagram as shown in Fig. 1. Here, it is assumed that the growth function is the standard logistic growth function, $G(x) = rx(1 - x/K)$. The analysis that generates Fig. 1 is reproduced in the supplementary materials.

We are mostly concerned with the stable manifold, which represents the optimal path of the fishery through (x, μ)-space. Any optimal path from an arbitrary starting point $x(0)$ starts on this manifold and converges to the steady state. The manifold can be interpreted as a function where for any x, μ is the corresponding shadow price. This is the function $\mu(x)$ that is found by solving Eq. (11). Obviously, $\mu(x)$ is also the derivative of the value function $V(x)$. Thus, the value function can be found by integrating $\mu(x)$ over $[0, x]$ and can be illustrated in Fig. 1 as the area below the stable manifold. For our purposes, the most important thing to note is that $\mu(x)$ appears to grow without limit as $x \to 0$.

3. The role of *ex post* shadow prices in the management of catastrophic risk—why we do not care (much)

Having argued that infinite shadow prices turn up endogenously, the question is then if we should be willing to use large resources to avoid a probability that we experience a state of the world with such a situation. The answer is no, and the reason can be found in the first-order conditions for optimization problems with catastrophes. Note that the arguments presented in this paper hold for *any* case where there is an infinite *ex post* shadow price and is not limited to the case where

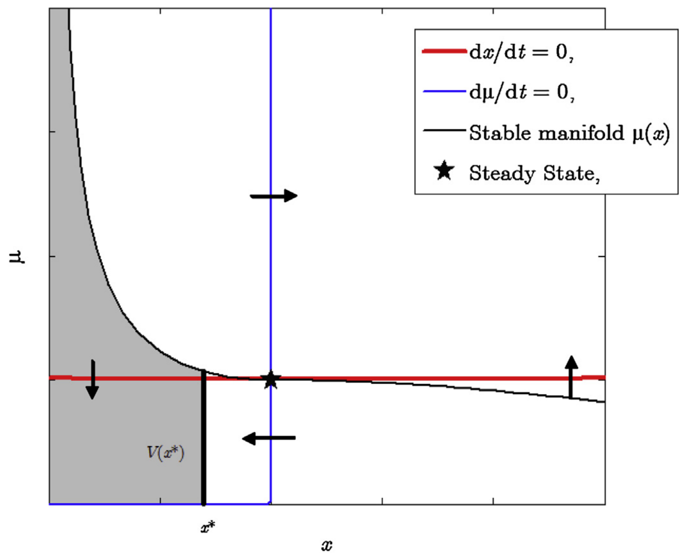

Fig. 1. Phase diagram for the simple fisheries model. The stable manifold shows combinations of μ and x along an optimal path and is the derivative of the value function. It is found by solving the differential equation in Eq. (11) using numerical techniques under the assumption that $G(x)=rx(1-x/K)$. The value function can be computed for any x^* by computing the area under $\mu(x)$ over the interval $[0, x^*]$, as illustrated by the shaded area. The parameter values $p=5$, $K=10$, $\rho=0.05$ and $r=1$ have been used in the plot.

we encounter division by zero as in Eq. (13). Thus, the arguments below also apply to cases where $F'_u(0,0)=\infty$ and any other circumstance that generates $\lim_{x\to 0}\mu(x)=\infty$.

We consider an optimal control problem with the notation from Eq. (1). The problem has two phases, one before and one after a catastrophe. The problem after the catastrophe is a simple deterministic control problem with value function $V(x)$ and shadow price $\mu(x)=V'(x)$. We impose that $\lim_{x\to 0}\mu(x)=\infty$. Without loss of generality, we assume $\mu(x)\geq 0$ so that the state variable is a desirable commodity. We specify a hazard process such that the hazard rate at any point in time is given by:

$$\lambda=\lambda(x) \tag{14}$$

Denote the point in time that the shock occurs as τ and the shock that occurs is that x experiences a jump given by $g(x)$. Thus, $x(\tau^+)=x(\tau^-)+g(x(\tau^-))$. Here, $x(\tau^-)=\lim_{t\uparrow\tau}x(t)$ and $x(\tau^+)=\lim_{t\downarrow\tau}x(t)$. The objective is then to solve the following problem

$$\begin{aligned}&\max_{u\in U\subseteq\mathbb{R}} E\left(\int_0^{+\infty}F(x,u)e^{-\rho t}\mathrm{d}t\right)\\ &\text{s.t}\dot{x}=f(x,u),\ x(0)=x_0\\ &x(\tau^+)=x(\tau^-)+g(x(\tau^-))\\ &\lim_{\Delta t\to 0}\frac{Pr(\tau\in(t,t+\Delta t)|\tau>t)}{\Delta t}=\lambda(x)\end{aligned} \tag{15}$$

Using optimality conditions found in Seierstad (2009), p. 130, we can phrase the optimality conditions in the terms of a current value risk-augmented Hamiltonian.

$$H(u,x,\mu,z)=F(x,u)+\mu_r f(u,x)+\lambda(x)(V(x+g(x))-z) \tag{16}$$

The optimality conditions are discussed in detail in Nævdal (2006). Here, z is the value of the value function at time t conditional on the catastrophe not having occurred over the time interval $[0, t]$, and $V(x+g(x))-z$ is thus the cost of the catastrophe and, here, is assumed to be negative. The variable μ_r is the shadow price prior to the catastrophe occurring. Denoting the optimal control as u^*, the optimality conditions from Seierstad (2009) are:

$$u*=\underset{u}{\operatorname{argmax}}H(u,x,\mu_r,z) \tag{17}$$

$$\begin{aligned}&\dot{\mu}_r=\rho\mu_r-F'_x(x,u*)-\mu_r f'_x(x,u*)\\ &-\lambda(x)(\mu_r-\mu(x+g(x))(1+g'(x)))-\lambda'_x(x)(z-V(x+g(x)))\end{aligned} \tag{18}$$

$$\dot{z} = \rho z - F(x, u*) - \lambda(x)(V(x + g(x)) - z) \tag{19}$$

In addition, if $x^*(t)$ is the optimal path, a transversality condition that $\lim_{t\to\infty} \mu_r(t)e^{-\rho t}(x(t) - x^*(t)) \geq 0$ must hold for all admissible $x(t)$. Recall that one interpretation of $\dot{\mu}$ is that for a given value of x, abs($\dot{\mu}$) is a measure of how urgently we would like to move to other locations in state space. In the present context, if $\dot{\mu}$ is smaller than 0, this implies that we are at a location in state space where the probability and/or the consequences of a catastrophe are severe and we wish to move to safer grounds.

Formally, the possibility of disaster affects the optimality conditions through Eqs. (18) and (19). Eq. (18) is the differential equation for the co-state variable. We can use this equation to decompose the effect of catastrophic risk into the effect of the hazard rate and differences in the *ex post* and *ex ante* solutions in a manner similar to, for example, that described by de Zeeuw and Zemel (2012) or Tsur and Zemel (1998). The first three terms of the right hand side Eq. (18) are the same as in a deterministic control problem. The two remaining terms are:

$$-\underbrace{\lambda(x)\left(\mu_r - \mu(x + g(x))\left(1 + g'(x)\right)\right)}_{\text{Effect caused by different ex post and ex ante shadow price}} - \underbrace{\lambda'_x(x)(z - V(x + g(x)))}_{\text{Effect caused by different ex post and ex ante value functions}} \tag{20}$$

These two terms are the modification that catastrophic risk requires to the differential equation for μ_r. The term $-\lambda(x)(\mu_r - \mu(x + g(x))(1 + g'(x)))$, which is the probability of a disaster at any point in time multiplied by a weighted difference between the *ex post* and *ex ante* shadow price, determines how $\dot{\mu}_r$ respond to differences in the shadow price before and after a catastrophe. The last term, $-\lambda'_x(x)(z - V(x + g(x)))$, shows how $\dot{\mu}_r$ responds to differences in the absolute levels of welfare before and after a catastrophe. Note that the only place where *ex post* shadow price $\mu(x)$ enters optimality conditions is in Eq. (18), which includes the term $\mu(x + g(x))(1 + g'(x))$.

Now, let us consider the consequences of a catastrophe. The straightforward way to do so is to stipulate that x collapses to zero. Thus, $g(x) = -x(\tau^-)$ and $g'(x) = -1$. Examining $\mu(x + g(x))(1 + g'(x))$ shows us that $\mu(x(\tau^-) - x(\tau^-))(1 - 1)$ is of the form "$\infty \times 0$." We therefore examine a limit where we let $g(x) = -\alpha x(\tau^-)$ and $g'(x) = -\alpha$ and study the behavior as $\alpha \to 1$. We have three possible outcomes:

$$\lim_{\alpha\to 1} \mu\left(x\left(\tau^-\right) - \alpha x\left(\tau^-\right)\right)(1 - \alpha) = \begin{cases} \infty \\ d \in \mathbb{R}/0 \\ 0 \end{cases} \tag{21}$$

If the expression in Eq. (21) goes to infinity, then we would clearly expend substantial resources to avoid the probability of a shock. If it is some constant d, we would be concerned, but finitely so. If it goes to zero, then the derivative of the value function at $x = 0$ does not affect the optimal program. In a few cases we can evaluate Eq. (21) directly and find that the limit reduces to zero. A more general result may be found using the fact that for any differentiable function $h(\cdot)$ defined in a neighborhood of zero, it holds that if $h(0) = 0$, then:

$$\lim_{y\to 0} \frac{h(y)}{y} = \lim_{y\to 0} h'(y) = h'(0) \tag{22}$$

Eq. (22) follows directly from the definition of the derivative. The last equality only holds if $h'(0)$ exists. Only the first equality is required below. Using this we result, we can state Proposition 3.

Proposition 3. Assume that the value function $V(x)$ satisfies $V(0) = 0$ and that $\lim_{x\to 0} \mu(x) = \infty$; then, $\lim_{\alpha\to 1} \mu((1 - \alpha)x(\tau^-))(1 - \alpha) = \lim_{y\to 0} \mu(yx(\tau^-))y = 0$ where $x(\tau^-) > 0$.

Proof. Let $h(y) = V(yx(\tau^-))$. Then, $h'(y) = V'(x(\tau^-)y)x(\tau^-)$. From Eq. (22), it follows that

$$\lim_{y\to 0} \left(\frac{h(y)}{y} - h'(y)\right) = 0 \Leftrightarrow \lim_{y\to 0} \left(h(y) - h'(y)y\right) = 0 \tag{23}$$

It follows that $\lim_{y\to 0} h'(y)y = \lim_{y\to 0} V'(x(\tau^-)y)x(\tau^-)y = \lim_{y\to 0} \mu(yx(\tau^-))x(\tau^-)y = 0$. However, then $\lim_{y\to 0} \mu(yx(\tau^-))y = 0$.. □

The assumption that $V(0) = 0$ may seem restrictive. However, as long as the value function is finite, we can add some constant to the integrand in Eq. (15) so that it becomes $(F(x, u) + b)e^{-\rho t}$ and always ensure that $V(0) = 0$ without altering the solution to Eq. (15) or the magnitude of $\mu(x)$.

Proposition 3 is quite helpful because it applies to a large class of problems including not only the type covered by Propositions 1 and 2 but also the type where the shadow price goes to infinity because the utility function satisfies Inada-conditions. Regardless of why the *ex post* shadow price is infinite at $x = 0$, it does not affect optimality conditions.

Proposition 3 only applied to cases where the *ex post* value function was bounded at $x = 0$. General results on the case where the *ex post* value function takes the value $-\infty$ are hard to obtain, but in this case, it is the effect of the infinite cost of the disaster that drives the optimal solution. However, if we assume that the value function is, for example, logarithmic, that is $V(x) = A + B \times \ln(x)$, as described by Brock and Mirman (1972), we can calculate that:

$$\mu\left((1-\alpha)x\left(\tau^{-}\right)\right)(1-\alpha) = \frac{B(1-\alpha)}{(1-\alpha)x(\tau^{-})} = \frac{B}{x(\tau^{-})} > 0 \tag{24}$$

Thus, with a logarithmic value function, the optimal solution to the pre-catastrophe problem depends to some extent on the post-disaster shadow price, but in a limited manner unless *x prior* to a shock happens to be very close to zero. Note that this implies that even if the total willingness to pay to avoid such a situation is infinity, the effect of the infinite *shadow price* is bounded.

From Eq. (20), one can also see that that if the hazard rate is endogenous, then the post-catastrophe value function $V(0)$ plays a role. For example, if $V(0)$ is negative with a large absolute value and $\lambda'_x(x) > 0$, then $\dot{\mu}_r$ will change very quickly so that x is driven to regions where risk is reduced or the consequences of a catastrophe are less dramatic. However, in this case, it is not the shadow price after a catastrophe that drives the change; rather, it is the difference in the levels of utility before and after a catastrophe that is important. Also note that if the hazard rate is exogenous so that $\lambda'_x(x) = 0$, then the magnitude of $V(0)$ does not matter either. In fact, it is in itself interesting that optimal policies prior to catastrophes with exogenous risk only depend on differences in *ex post* and *ex ante* shadow prices and not by differences in absolute levels of welfare. This is because if risk is exogenous, the disaster may happen or not, but there is no way to affect this, but by choosing *ex ante* levels of state variables, we can affect our preparedness for the post event situation as the levels of *ex ante* variables also determine the *ex post* levels of these variables. The catastrophe analyzed in this paper is an exception to this rule, as a total collapse regardless of the value of *ex ante* state variables implies that one cannot affect preparedness.

4. Summary and conclusion

We have shown that value functions with infinite derivatives for critical values may turn up endogenously in problems where neither the integrand nor the growth function specified in the problem have infinite derivatives anywhere on their domains. The explanation is that regenerative processes may exhibit an infinite marginal value if the intrinsic growth rate is larger than the discount rate. This mechanism has been demonstrated for some specific functional forms, but one may argue that this possibility should be checked whenever growth in the state variable is zero for values of the stock outside of the steady state.

However, the infinite derivatives property does not exclude that it may be optimal to end in such a state, only that it will never be optimal to do so in finite time as an incremental process. We show this by demonstrating that in the particular case where a shock consists of a total collapse in the state variable to zero, the shadow price of a resource does not affect optimality conditions for management before the shock occurs. Therefore, if the system is subject to catastrophic shocks, it may very well be optimal that we accept a positive probability of ending up permanently in a state where the corresponding value function has an infinite derivative.

Appendix A. The solution to Eq. (11).

The solution to Eq. (11).

The solution in Eq. (10) can be found using the following calculations:
Dividing Eq. (11) by μ and integrating over $[x, x*]$ gives:

$$\int_{x}^{x*}\frac{1}{\mu(x)}\frac{d\mu}{dy}dy = \int_{x}^{x*}\frac{\rho}{G(y)}dy - \int_{x}^{x*}\frac{G'(y)}{G(y)}dy$$

$$-\int_{\mu(x*)}^{\mu(x)}\frac{1}{\mu}d\mu = \int_{x}^{x*}\frac{\rho}{G(y)}dy - (\ln(G(x*)) - \ln(G(x)))$$

$$\ln\left(\frac{\mu(x)}{\mu(x*)}\right) = -\int_{x}^{x*}\frac{\rho}{G(y)}dy + \ln\left(\frac{G(x*)}{G(x)}\right)$$

$$\frac{\mu(x)}{\mu(x*)} = \frac{G(x*)}{G(x)}\exp\left(-\int_{x}^{x*}\frac{\rho}{G(y)}dy\right)$$

Inserting for $\mu(x*) = p$ and rearranging gives the expression for $\mu(x)$ in Eq. (10):

$$\mu(x) = \frac{pG(x*)}{G(x)} \exp\left(-\int_x^{x*} \frac{\rho}{G(y)}dy\right)$$

Calculating the expression in Eq. (12).

Applying L'Hôpital's rule to Eq. (10) yields

$$\lim_{x\to 0}\mu(x) = pG(x*)\frac{\lim\limits_{x\to 0}\frac{d}{dx}\left(\exp\left(-\int_x^{x*}\frac{\rho}{G(y)}dy\right)\right)}{\lim\limits_{x\to 0}G'(x)}$$

$$= pG(x*)\frac{\lim\limits_{x\to 0}\exp\left(-\int_x^{x*}\frac{\rho}{G(y)}dy\right)\frac{\rho}{G(x)}}{\lim\limits_{x\to 0}G'(x)}$$

$$= pG(x*)\lim_{x\to 0}\frac{\rho}{G'(x)} \times \lim_{x\to 0}\frac{\exp\left(-\int_x^{x*}\frac{\rho}{G(y)}dy\right)}{G(x)}$$

$$= \frac{\rho}{G'(0)}\lim_{x\to 0}\frac{pG(x*)}{G(x)}\exp\left(-\int_x^{x*}\frac{\rho}{G(y)}dy\right)$$

The last line implies that:

$$\lim_{x\to 0}\mu(x) = \frac{\rho}{G'(0)}\lim_{x\to 0}\mu(x)$$

Appendix B. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.jebo.2016.04.021.

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