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# Equity and Efficiency in an Overlapping Generation Model

Tanguy Isaac · Paolo G. Piacquadio

**Abstract** The paper addresses *intergenerational* and *intragenerational* equity in an overlapping generation economy. We aim at defining an egalitarian distribution of a constant stream of resources, relying on ordinal non-comparable information on individual preferences. We establish the impossibility of efficiently distributing resources while treating equally agents with same preferences that belong to possibly different generations. We thus propose an egalitarian criterion based on the equal-split guarantee: this requires all agents to find their assigned consumption bundle at least as desirable as the equal division of resources.

**Keywords** Intergenerational equity · Intragenerational equity · No-envy · Equal-split guarantee · Allocation rules

JEL Classification: D61, D63, D91.

## 1 Introduction

We address *intra*- and *inter*-generational justice in an infinite-horizon overlapping generation economy. The ethical problem is to share a constant windfall of resources across infinite many generations, each consisting of a finite and constant number of 2-periods living agents. How would an egalitarian ethical observer assign the available goods, relying on ordinal and non-comparable information about preferences?

In the static counterpart, i.e. with only one period, the model is equivalent to the “classical problem of fair division,” in which a social endowment is to be distributed among heterogeneous agents. In such a setting, several appealing solutions have been proposed: among them, the equal-division Walrasian assigns to each agent her preferred

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bundle from a budget set defined by the equal-split of resources and the Walrasian prices.<sup>1</sup> This paradigmatic distribution of resources satisfies three appealing properties:

1. it is not possible to make someone better-off without making someone else worse-off;
2. each agent prefers the bundle she is assigned to the equal share of the social endowment;
3. no agent would be better-off with the bundle assigned to someone else.

The first property is Pareto efficiency. The second and third properties are respectively the “equal-split guarantee,” proposed by Steinhaus (1948), and the “no-envy” criterion, introduced by Foley (1967) and Kolm (1972). Unfortunately, no corresponding distribution of resources can be found in the dynamic setting.

We establish a strong tension between efficiency and equity. When multiple generations are considered, it is not possible to distribute resources efficiently and together ensure that same-preference agents be treated alike.<sup>2</sup> This axiom is much weaker than *no-envy* as it remains silent about welfare comparisons between agents with different preferences. Endowing each period with the same amount of available resources goes in the direction of making *efficiency* and equity compatible, but is not sufficient to avoid a clash between the two.<sup>3</sup> Agents belonging to different generations might still face a time specific economic environment: this directly depends on the type of agents living at the same time (both the members of the same generation and the overlapping agents), but also on the type and order of all the previous and next generations.

A way to avoid such difficulties is to *efficiently* distribute goods satisfying the equity requirement of “no-domination,” introduced by Thomson (1983). Also related and logically weaker than *no-envy*, this axiom requires that no agent be assigned less goods than some other agent. The ethical appeal of such allocations is however undermined by the impossibility of securing each agent with an arbitrarily small lower bound on well-being, when selecting allocations that satisfy *efficiency* and *no-domination*.

An appealing alternative is to combine *efficiency* with an appropriate version of the *equal-split guarantee*. Resources should be *efficiently* shared in such a way that each agent finds its assigned bundle at least as desirable as an equal share of resources (the

<sup>1</sup> For an overview of this literature, see Moulin (1990) and Thomson (2011).

<sup>2</sup> This requirement of “equal treatment of equals” is a weak counterpart of anonymity for frameworks with no comparable information about preferences. In the main economic approach to intergenerational equity, each generation is described by a utility level and the objective is to define how to rank infinite vectors of these utilities (utility streams), based on appealing equity and efficiency conditions. The seminal contribution of Diamond (1965) showed a strong tension between efficiency and equity: a continuous and complete ranking of these utility streams cannot satisfy both Pareto efficiency and “finite anonymity”. The anonymity concept expresses equal concern for all generations by requiring the ranking to be invariant to permutations of the utilities of a finite number of generations. For a survey of this literature, see Asheim (2010).

<sup>3</sup> When periods are differently endowed in terms of resources, treating agents equally might impede to distribute all the resources available during affluent times. This negative conclusion is called “leveling down objection” and opposes equity with efficiency. Parfit (1997) discusses the differences between a *pure* egalitarian view, that cares only about equality, and a *pluralist* egalitarian view, that aims at combining equity with efficiency. For an illustration consider the two following alternatives: in A everyone gets 10 units; in B half of the agents get 10, half get 20. According to the first view A is better but incurs in the leveling down; according to the second view B is better, but needs to compromise equity. Our strategy is to consider a model with the most favorable conditions in terms of compatibility between efficiency and equity, i.e. with constant flow of resources. An extension to non constant resources is introduced in Section 4.

multiplicity of such choices is brought by the overlapping generation structure). Interestingly, each such allocation ensures that, at least to some extent, also *no-domination* holds. Maximizing the degree to which *no-domination* holds leads to the reference equal split of resources that shares the endowment equally between young and old agents.

In Dubey (2013), a recent contribution related to ours, the tension between *efficiency* and *no-envy* is of a different nature. In a setting with a single good, one agent per generation, and time invariant preferences, the impossibility of treating agents equally derives from the exogenous quantity that the first old generation – the generation that is old in the first period – is assigned to consume when young. When this level is sufficiently high, efficiency makes the first generation better-off than later ones; when this is sufficiently low, “fair” distributions become possible. In the last case, there exist efficient alternatives that assign a more desirable bundle to all generations but the first one; such alternatives can, however, not be selected since these would violate *no-envy*. In a similar framework with population growth, Shinotsuka et al (2007) discusses the appeal of different concepts of *no-envy* and their relation with the exogenous growth rate of the population.<sup>4</sup> Compared to their contributions, we relax the equity requirement of the first old generation by sterilizing the role of their consumption when young and avoid their difficulties. Ethical difficulties persist due to the richer setting, with heterogeneous agents and many commodities.

In a one-commodity overlapping generation model with intergenerational transfers, Fleurbaey (2007) singles out a welfare criterion that endogenously determines a comparable well-being measure. While the scope of that contribution is wider than ours, egalitarian distribution of resources are focal and partial or complete ordering can be constructed for each allocation rule (see Fleurbaey and Maniquet, 2011). In a multi-dimensional commodity model with intergenerational transfers, but representative one-period-living agents, a different tension arises: *equal-treatment of equals* and *no-domination* can be singularly combined with *efficiency*, but not jointly (Piacquadio, 2014). Finally, in an overlapping generation setting with comparable and additively separable utilities, Quiggin (2012) has shown that *efficiency* and within-period utilitarianism requires adopting the undiscounted utilitarian criterion across generations.

The paper is organized as follows. In Section 2, we introduce the model. In Section 3, we present the axiomatic analysis and discuss the results. In Section 4 we allow for non-constant resource endowments: this confirms the clash between equity axioms related to *no-envy* and efficiency; conversely, the egalitarian solution based on the equal-split guarantee extends. Concluding remarks are contained in Section 5.

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<sup>4</sup> The authors build on three extensions of the *no-envy* criterion, introduced by Suzumura (2002) to fit the dynamic overlapping generation framework. The “no-envy in lifetime consumptions” holds when no agent finds the bundle assigned to some other agent more preferable than the own; the “no-envy in overlapping consumptions” holds when at each period no agent finds preferable the consumption bundle assigned for consumption at that period of some other young or old agent; the “no-envy in the lifetime rate of return” requires to equalize agent’s welfare measured according to the concept of lifetime rate of return, due to Cass and Yaari (1966). As in Dubey (2013), we interpret *no-envy* in terms of lifetime consumptions.

## 2 The model

We consider a two-periods overlapping generation model. Let  $t \in \mathbb{N} \equiv \{0, 1, 2, \dots\}$  be the time index. For each  $t \in \mathbb{N}$ , generation  $t$  is the cohort of agents that are young at time  $t$ ; it consists of a constant (finite) number of agents  $I$ .<sup>5</sup>

The first period is  $t = 1$ ; this brings forth the *Adam generation* (generation 0), consisting of the  $I$  “first-old agents”: the model describes their preferences and allocations only at time 1. Therefore, in each period  $t \in \mathbb{N}_{>0} \equiv \{1, 2, \dots\}$ , there are  $I$  young agents of generation  $t$  and  $I$  old agents of generation  $t - 1$ .

In each period there are  $L$  infinitely divisible and privately appropriable goods indexed by  $l \in \{1, \dots, L\}$ ; for notational simplicity, we also identify by  $L$  the set of goods. Each agent  $(i, t) \in I \times \mathbb{N}_{>0}$  is allocated a **consumption bundle**  $a_i(t) \equiv (c_i(t), d_i(t))$ , where **consumption when young** is  $c_i(t) \equiv (c_i^1(t), \dots, c_i^L(t)) \in \mathbb{R}_+^L$  and **consumption when old** is  $d_i(t) \equiv (d_i^1(t), \dots, d_i^L(t)) \in \mathbb{R}_+^L$ . The agents of the Adam generation are only assigned a consumption when old, i.e.  $a_i(0) \equiv d_i(0) \equiv (d_i^1(0), \dots, d_i^L(0)) \in \mathbb{R}_+^L$  for each  $i \in I$ . Thus,  $(c(t), d(t)) \in \mathbb{R}_+^{2LI}$  specifies the consumption bundles of agents of generation  $t \in \mathbb{N}_{>0}$  and  $d(0) \in \mathbb{R}_+^{LI}$  specifies the consumption bundles of agents of generation 0. We denote an **allocation** by  $a$ , i.e. a list of consumption bundles for each agent  $(i, t) \in I \times \mathbb{N}$ . Agent’s preferences are self-centered, i.e. defined on own consumption; these are denoted by  $\succsim_{i,t}$  and are assumed to be complete, transitive, continuous, convex, and strictly monotonic.<sup>6</sup> Let  $\succ_{i,t}$  and  $\sim_{i,t}$  denote the asymmetric and symmetric counterpart of  $\succsim_{i,t}$ .

The available resources are a constant stream that, without loss of generality, is normalized to 1 for each good. We assume that no transfer across time is possible: resources available at  $t \in \mathbb{N}_{>0}$  can be allocated only to agents living at  $t$  (either young or old). An allocation  $a$  is **feasible** if  $\sum_{i \in I} d_i^l(t-1) + \sum_{i \in I} c_i^l(t) \leq 1$  for each good  $l \in L$  and for each period  $t \in \mathbb{N}_{>0}$ . Let  $A$  denote the **set of feasible allocations**.

An **economy** is defined by the specification of each agent’s preferences:  $E \equiv (\{\succsim_{i,t}\}_{(i,t) \in I \times \mathbb{N}})$ . Let  $\mathcal{E}$  denote the set of economies that satisfy the above assumptions.

## 3 The axiomatic analysis

Following the approach of fair allocation theory, we aim at identifying a subset of ethically appealing ways to distribute the available resources. Let an (allocation) **rule** be a correspondence  $\psi : \mathcal{E} \rightarrow 2^A \setminus \{\emptyset\}$ , which selects a (non-empty) subset of feasible allocations for each economy in the domain. The ethical appeal of a rule is judged by the axioms it satisfies. We will adopt the following convention: a rule  $\psi$  satisfies an axiom if and only if each allocation selected by the rule (for each economy in the domain) satisfies it; we thus define axioms in terms of properties of allocations only.

The first axiom requires that there is no other feasible allocation that is at least as desirable for each individual and strictly more desirable for some.

<sup>5</sup> With a slight abuse of notation, we denote by  $I$  both for the set of agents and its cardinality.

<sup>6</sup> The convexity requirement is not necessary for our existence result. Nevertheless, we do assume this restriction to avoid that the difficulties with alternative equity conditions originate from it. All impossibility results extend to the case of strictly convex preference.

**Axiom.** For each  $E \in \mathcal{E}$ , an allocation  $a \in A$  is (Pareto) **efficient** for  $E$  if there is no allocation  $a' \in A$  s.t. for each agent  $(i, t) \in I \times \mathbb{N}$ ,  $a'_i(t) \succsim_{i,t} a_i(t)$  and for some agent  $(i, t) \in I \times \mathbb{N}$ ,  $a'_i(t) \succ_{i,t} a_i(t)$ .

### 3.1 The equal-split guarantee

The next axiom imposes an ordinal lower bound on the welfare of individuals: each agent should find its assigned bundle at least as desirable as at the equal division of the available resources. Differently from the static setting, however, the equal division of resources is not uniquely identified. Let  $(\bar{c}, \bar{d}) \in \mathbb{R}_+^{2LI}$  satisfy  $\bar{c}^l + \bar{d}^l = \frac{1}{I}$  for each good  $l \in L$ ; then, an **equal-split allocation**  $\bar{a} \in A$  is such that for each agent  $(i, t) \in I \times \mathbb{N}_{>0}$ ,  $\bar{a}_i(t) = (\bar{c}, \bar{d})$  and for each agent  $i \in I$ ,  $\bar{a}_i(0) = \bar{d}$ . Clearly, for each such  $(\bar{c}, \bar{d})$ , a different equal-split allocation arises. Let  $A^s \subset A$  be the set of equal-split allocations.

**Axiom.** For each  $\bar{a} \in A^{es}$  and each  $E \in \mathcal{E}$ , an allocation  $a \in A$  satisfies  **$\bar{a}$ -equal-split guarantee** for  $E$  if for each agent  $(i, t) \in I \times \mathbb{N}_{>0}$ ,  $a_i(t) \succsim_{i,t} \bar{a}_i(t) = (\bar{c}, \bar{d})$  and for each agent  $i \in I$ ,  $a_i(0) \succsim_{i,0} \bar{a}_i(0) = \bar{d}$ .

The following result is immediate and the proof is omitted.

**Theorem 1** *Let  $\bar{a} \in A^{es}$ . On the domain  $\mathcal{E}$ , there exist a rule that satisfies efficiency and  $\bar{a}$ -equal-split guarantee.*

The allocations selected by a rule satisfying *efficiency* and  *$\bar{a}$ -equal-split guarantee* might not be very egalitarian. Among those allocations it is possible that all but one agents of a generation are assigned a bundle that is just as desirable as the equal split  $(\bar{c}, \bar{d})$ , while the one being assigned a very large surplus. Hoping to avoid such inequalities, we next investigate to which extent it is possible to satisfy *no-envy*, requiring that each agent finds its assignment as desirable as that of any other agent.

### 3.2 Equal treatment of equals

The next axiom is logically weaker and ethically different from *no-envy*: it requires any two agents to be treated alike whenever they share the same preferences. The ethical observer should not discriminate agents based on the time they live in, i.e. in the proximate or distant future, as soon as they are indistinguishable in terms of personal traits. It is thus related to a key axiom of the utility streams literature, “finite anonymity,” requiring the evaluation of two alternatives be independent of reshuffling the utilities achieved by a finite number of them (see Fn. 2).

**Axiom.** For each  $E \in \mathcal{E}$ , an allocation  $a \in A$  satisfies **equal treatment of equals** if for each pair of agents  $(i, t), (\iota, \tau) \in I \times \mathbb{N}_{>0}$ ,  $\succsim_{i,t} = \succsim_{\iota,\tau}$  implies that  $a_i(t) \sim_{i,t} a_\iota(\tau)$ .

This requirement is however not compatible with *efficiency*, as the next result shows.

**Theorem 2** *On the domain  $\mathcal{E}$ , no rule satisfies Pareto efficiency and equal treatment of equals.*

*Proof* We construct an economy  $E \in \mathcal{E}$  for which no allocation satisfies *efficiency* and *equal treatment of equals*.

Let  $L = 2$ . Agents are of two kinds,  $\alpha$  and  $\beta$ , with preferences represented by the following functions:

$$\begin{aligned} U_\alpha(c^1, c^2, d^1, d^2) &= \gamma c^1 + c^2 + \gamma d^1 + d^2 \\ U_\beta(c^1, c^2, d^1, d^2) &= c^1 + \gamma c^2 + d^1 + \gamma d^2 \end{aligned}$$

Let generations  $t \in [1, 9]$  be such that: generations  $t \in [1, 2]$  consist of two  $\alpha$  agents; generations  $t \in [3, 7]$  consist of an  $\alpha$  and a  $\beta$  agent; generations  $t \in [8, 9]$  consist of two type  $\beta$  agents. Set  $\gamma > 15$ .

By *equal treatment of equals*, the  $\alpha$  agents of generations  $t \in [1, 2]$  should achieve the same utility level. The same is true for the agents  $\beta$  of generations  $t \in [8, 9]$ . Easy computation shows that the highest utility they can reach (both agents  $\alpha$  and  $\beta$  of generations  $t \in \{1, 2, 8, 9\}$ ) is  $U^{max} = \frac{3(\gamma+1)}{4}$ . The *Pareto efficient* distribution of the remaining resources implies that at least one agent ( $\alpha$  or  $\beta$ ) of generations  $t \in [3, 7]$  achieves a larger utility than  $U^{min} = \frac{4}{5}\gamma$ . As  $U^{min} > U^{max}$  when  $\gamma > 15$ , there is no distribution of resources for which *efficiency* and *equal treatment of equals* are jointly satisfied.  $\square$

Same-preference agents belonging to different generations might face very different environments, for which they cannot be fully compensated. The social cost of assigning a bundle to an agent depends on the amount available, but also (and crucially so) on the preferences of agents of the same generation, on the preferences of previous and next generations, and, indirectly, on all agent's preferences. Each period's social endowment is assumed constant. Nevertheless, circumstances might differ due to the social interaction between heterogeneous agents, resulting in different relative scarcity of goods: if such differences are too large, it might not be possible to treat same-preference agents equally. In Section 4, we explore the case in which resources can be costly transferred across time; however, the clash cannot be avoided unless transfers are free.

### 3.3 No-domination

This impossibility result is overcome when preferences have a limited role in defining how resources should be equitably distributed. The next result shows that it is possible to combine *efficiency* with another well-known axiom of equity, introduced by Thomson (1983) and Moulin and Thomson (1988). It requires that no agent is assigned at least as much of all goods as, and more of at least one good than, some other agent. This axiom excludes allocations that by being strictly greater than one other would imply envy for any possible preferences of the agents.

**Axiom.** An allocation  $a \in A$  satisfies **no-domination** if for each pair of agents  $(i, t), (\iota, \tau) \in I \times \mathbb{N}_{>0}$ ,  $a_i(t) \not\geq a_\iota(\tau)$  and for each pair of agents  $i, \iota \in I$ ,  $a_i(0) \not\geq a_\iota(0)$ .

**Theorem 3** *On the domain  $\mathcal{E}$ , there exists a rule that satisfies efficiency and no-domination.*

*Proof* The proof is constructive. Let the set of allocations that assign goods exclusively to old agents be  $A^{old} \equiv \{a \in A \mid c_{i,t} = 0 \forall i \in I, \forall t \in \mathbb{N}_{>0}\} \subset A$ . Let  $a^* \in A^{old}$  be the most efficient allocation in  $A^{old}$ , i.e. there is no  $a' \in A^{old}$  such that for each  $(i, t) \in I \times \mathbb{N}$ ,  $a'_i(t) \succsim_{i,t} a^*_i(t)$  and for some  $(i, t) \in I \times \mathbb{N}$ ,  $a'_i(t) \succ_{i,t} a^*_i(t)$ . By contradiction, assume that  $a^*$  is not *efficient* for  $E$  (among all the feasible allocations  $A$ ). Then, there exists  $a'' \in A \setminus A^{old}$ , such that for each  $(i, t) \in I \times \mathbb{N}$ ,  $a''_i(t) \succsim_{i,t} a^*_i(t)$  and for some  $(i, t) \in I \times \mathbb{N}$ ,  $a''_i(t) \succ_{i,t} a^*_i(t)$ . Since  $a'' \notin A^{old}$  there is  $t \in \mathbb{N}$  for which  $\sum_{i \in I} \bar{c}_i(t) > 0_L$ .<sup>7</sup> By feasibility, this implies that  $\sum_{i \in I} d''_i(t-1) < 1_L$ . If  $\sum_{i \in I} c''_i(t-1) = \sum_{i \in I} c^*_i(t-1) = 0_L$ , agents of generation  $t-1$  would be given less at  $a''$  than at  $a^*$  ( $\sum_{i \in I} d''_i(t-1) < \sum_{i \in I} d^*_i(t-1) = 1_L$ ) and would contradict that  $a''$  is more efficient than  $a^*$ ; thus,  $\sum_{i \in I} c''_i(t-1) > 0_L$ . Iterating the argument backwards,  $\sum_{i \in I} d''_i(\tau) < 1_L$  for each  $\tau \leq t$ . Since agents of the Adam generation consume only when old and  $\sum_{i \in I} d''_i(0) < 1_L$ , there must be at least an agent  $i \in I$  for which  $a^*_i(0) \succ_{i,t} a''_i(0)$ . This contradiction guarantees that any *efficient* allocation in  $A^{old}$  is also *efficient* in  $A$ . It remains to prove that such *efficient* allocation exists and satisfies *no-domination*. This is an immediate consequence of the “budget constrained Pareto optimal method” by Moulin (1991). There always exist an *efficient* allocation  $\hat{a} \in A$  such that for each pair  $(i, t), (i, \tau) \in I \times \mathbb{N}$ ,  $\sum_{l \in L} \hat{d}_i^l(t) = \sum_{l \in L} \hat{d}_i^l(\tau) = \frac{L}{N}$ . Since each agent’s bundle lies on the same hyperplane, it satisfies *no-domination*.  $\square$

While the result seems compelling, the proof suggests that allocations that satisfy *no-domination* and *efficiency* might not be very egalitarian. The existence result is proven by assigning all the social endowment to consumption during old age, independently of how heavily agents discount consumption when old. This trick avoids overlapping consumptions and makes each period’s resource distribution problem independent of earlier and later allocation choices. As we will discuss in the next subsection, the ethical appeal of *no-domination* is severely undermined by the impossibility of guaranteeing any positive ordinal lower bound to welfare.

### 3.4 Combining no-domination and the equal-split guarantee

As both the  $\bar{a}$ -equal-split guarantee and *no-domination* are compatible with *efficiency*, we shall now investigate to which extent these equity axioms can be jointly satisfied.

Unfortunately *no-domination*, together with *efficiency*, does not allow choosing allocations such that each agent is better-off than she would be by consuming any *arbitrarily small* bundle of goods. This negative result is stronger than showing that there exists no egalitarian solution that combines *efficiency* with *no-domination* and the  $\bar{a}$ -equal-split guarantee.

We shall weaken the  $\bar{a}$ -equal-split guarantee axiom in two respects. First, we require that each agent find its assigned consumption at least as desirable as a fraction  $\varepsilon \in (0, 1]$  of the equal-split allocation  $\bar{a} \in A^{es}$ .<sup>8</sup> Second, we allow to adjust the choice of  $\bar{a}$  to the particular situation at hand, provided this allocation is strictly positive.

<sup>7</sup> We denote by  $0_L$  and  $1_L$  the  $L$ -dimensional vector of zeros and ones.

<sup>8</sup> When  $\varepsilon = 1$ , the axiom of  $\bar{a}$ -equal-split guarantee is obtained; as  $\varepsilon$  decreases the condition becomes weaker and weaker; at the limit for  $\varepsilon = 0$ , it is vacuous. This requirement is similar to the  $\varepsilon$  version of “individual rationality” introduced by Moulin and Thomson (1988): their axiom requires each agent to be at least as well-off as when consuming a bundle that is the  $\varepsilon$ -share of the aggregate available resources. The idea of introducing a parametrization in the



**Axiom.** Let  $\varepsilon \in (0, 1]$ . For each  $E \in \mathcal{E}$ , an allocation  $a \in A$  satisfies  $\varepsilon$ -**equal-split guarantee** if there exist  $\bar{a} \in A^{es}$  with  $\bar{a} \gg 0$  such that for each agent  $(i, t) \in I \times \mathbb{N}_{>0}$ ,  $a_i(t) \succsim_{i,t} \varepsilon \bar{a}_i(t) = \varepsilon(\bar{c}, \bar{d})$  and for each agent  $i \in I$ ,  $a_i(0) \succsim_{i,0} \varepsilon \bar{a}_i(0) = \varepsilon \bar{d}$ .

**Theorem 4** *Let  $\varepsilon \in (0, 1]$ . On the domain  $\mathcal{E}$ , no rule satisfies efficiency, no-domination and the  $\varepsilon$ -equal-split guarantee.*

*Proof* See Appendix A.

Interestingly, the symmetrical argument involving an “epsilon” variant of no-domination does not hold. Given *efficiency*, the  $\bar{a}$ -equal-split guarantee is compatible with, and actually implies, a certain degree of *no-domination*. As before, we measure this degree by a parametric version of *no-domination*: no agent should be assigned a bundle that is dominated by the  $\varepsilon$ -part of the bundle of any other agent.

**Axiom.** Let  $\varepsilon \in [0, 1]$ . For each  $E \in \mathcal{E}$ , an allocation  $a \in A$  satisfies  $\varepsilon$ -**no-domination** if for each pair of agents  $(i, t), (\iota, \tau) \in I \times \mathbb{N}_{>0}$ ,  $\varepsilon a_i(t) \not\succ a_\iota(\tau)$  and for each pair of agents  $i, \iota \in I$ ,  $\varepsilon a_i(0) \not\succ a_\iota(0)$ .

**Theorem 5** *Let  $\bar{a} \in A^{es}$ . On the domain  $\mathcal{E}$ , if a rule satisfies Pareto efficiency and the  $\bar{a}$ -equal-split guarantee then it satisfies  $\varepsilon$  no-domination for each  $\varepsilon \leq \min[\min_l \bar{c}^l, \min_l \bar{d}^l]$ .*

*Proof* If  $(\bar{c}, \bar{d})$  has some zero components,  $\varepsilon = 0$  and  $\varepsilon$ -no-domination is trivially satisfied. Thus, let  $\bar{a} \in A^{es}$  be such that  $(\bar{c}, \bar{d}) \gg 0_{2L}$  and let  $\varepsilon \leq \min[\min_l \bar{c}^l, \min_l \bar{d}^l]$ . By contradiction, assume that  $\varepsilon$ -no-domination is violated. Then there exist  $(i, t), (\iota, \tau) \in I \times \mathbb{N}_{>0}$  (or  $i, \iota \in I$ ) such that  $(c_i(t), d_i(t)) \ll \varepsilon(c_\iota(\tau), d_\iota(\tau))$  (resp.  $d_i(0) \ll \varepsilon d_\iota(0)$ ).

By the *equal-split guarantee*,  $(c_i(t), d_i(t)) \succsim_{i,t} (\bar{c}, \bar{d})$  (resp.  $d_i(0) \succsim_{i,0} \bar{d}$ ); as  $\varepsilon \leq \min[\min_l \bar{c}^l, \min_l \bar{d}^l]$ , we have that for all bundles  $(c_\iota(\tau), d_\iota(\tau)) \in [0, 1]^{2L}$ ,  $\varepsilon(c_\iota(\tau), d_\iota(\tau)) \ll (\bar{c}, \bar{d})$ . By preference monotonicity, we derive a contradiction:  $(c_i(t), d_i(t)) \succsim_{i,t} (\bar{c}, \bar{d}) \succ_{i,t} \varepsilon(c_\iota(\tau), d_\iota(\tau)) \succ_{i,t} (c_i(t), d_i(t))$  (resp.  $d_i(0) \succsim_{i,0} \bar{d} \succ_{i,0} \varepsilon d_\iota(0) \succ_{i,0} d_i(0)$ ).  $\square$

As the result states, the upper bound for the degree  $\varepsilon$  of  $\varepsilon$ -no-domination satisfied by the selected allocations depends on the choice of the equal-split allocation  $\bar{a} \in A^{es}$ . When minimizing the possible inequalities due to domination of consumption bundles, a specific reference arises.<sup>9</sup> This is the **age-independent equal-split allocation**: this allocation is such that  $\bar{c}^l = \bar{d}^l = \frac{1}{2I}$  for each good  $l \in L$  and, thus,  $\bar{c} = \bar{d}$ . This result is formalized in the next corollary.

**Corollary 1** *On the domain  $\mathcal{E}$ , let a rule exist which satisfies efficiency,  $\bar{a}$ -equal-split guarantee with  $\bar{a} \in A^{es}$ , and  $\varepsilon$ -no-domination with  $\varepsilon \in [0, 1]$ . Then,  $\varepsilon$  is maximal when  $\bar{a}$  is the age-independent equal-split allocation.*

distributional criteria has been further exploited in the literature of fair allocations: similar criteria are defined, among others, in Thomson (1987) and Sprumont (1998); in the dynamic setting it has been adopted by Piacquadio (2014).

<sup>9</sup> We are indebted to Yves Sprumont for highlighting this point.

## 4 Extensions

In this section we show that we can relax the assumption about constant stream of resources without affecting the appeal of the defined egalitarian rule. Conversely, this will further increase the tension between *efficiency* and *no-envy* related equity concepts.

### 4.1 Non-constant resource streams

The assumption that resources are constant over time goes in the direction of giving a chance for the existence of an egalitarian criterion in the ordinal and non-comparable framework. When the stream of resources is time varying, the impossibility of transferring resources over time would impede any compensation between generations of agents that are differently endowed. This directly entails the incompatibility of *equal treatment of equals* (or of *no-domination*) with *Pareto efficiency* even with stationary preferences.

Interestingly, an appealing extension of the *equal-split guarantee* can be introduced for non-constant streams of resources. For each  $t \in \mathbb{N}_{>0}$ , let the vector of resources be  $\omega_t \equiv (\omega_t^1, \dots, \omega_t^L) \in \mathbb{R}_{++}^L$  such that  $\inf_{l \in L} \omega_t^l > \phi$  for some  $\phi > 0$ . This restriction guarantees that resources are bounded away from zero at any period  $t \in \mathbb{N}_{>0} \cup \{\infty\}$ . Let an **economy** be a list  $\bar{E} \equiv (\{\omega_t\}_{t \in \mathbb{N}_{>0}}, \{\succsim_{i,t}\}_{i \in I, t \in \mathbb{N}})$  and let  $\bar{\mathcal{E}}$  be the domain of overlapping generation economies with non-constant resources.

For each  $\bar{E} \in \bar{\mathcal{E}}$ , an allocation  $a$  is **feasible for  $\bar{E}$**  if  $\sum_{i \in I} d_i^l(t-1) + \sum_{i \in I} c_i^l(t) \leq \omega_t^l$  for each good  $l \in L$  and for each period  $t \in \mathbb{N}_{>0}$ . Let  $A(\bar{E})$  denote the set of feasible allocations for  $\bar{E}$ .

For each  $\bar{E} \in \bar{\mathcal{E}}$ , an allocation  $\bar{a} \in A(\bar{E})$  is a **lower-bound division of resources** if there is a pair  $\bar{c}, \bar{d} \in \mathbb{R}_{++}^L$  such that for each agent  $i \in I$  of generation  $t \in \mathbb{N}_{>0}$ ,  $\bar{a}_i(t) = (\bar{c}, \bar{d})$  and for each agent  $i \in I$  of generation 0,  $\bar{a}_i(0) = \bar{d}$ . Let  $A^{lb}(\bar{E}) \subset A(\bar{E})$  be the set of lower bound division of resources. As resources are bounded away from zero,  $A^{lb}(\bar{E})$  is non-empty. It is thus possible to select one of these allocations as the lower-bound for the egalitarian distribution problem:

**Axiom.** For each  $\bar{E} \in \bar{\mathcal{E}}$  and each  $\bar{a} \in A^{lb}(\bar{E})$ , an allocation  $a \in A$  satisfies  **$\bar{a}$ -lower-bound guarantee** if for each agent  $(i, t) \in I \times \mathbb{N}_{>0}$ ,  $a_i(t) \succsim_{i,t} \bar{a}_i(t) = (\bar{c}, \bar{d})$  and for each agent  $i \in I$ ,  $a_i(0) \succsim_{i,0} \bar{a}_i(0) = \bar{d}$ .

The possibility result of Theorem 1 thus extends: *efficiency* and the  $\bar{a}$ -lower-bound guarantee – for any choice of  $\bar{a}$  – are compatible.

### 4.2 Intertemporal resource transfers

We here show that even when transfers can be transferred across time, the clash between *efficiency* and *equal treatment of equals* cannot be avoided. Conversely the existence of a rule satisfying *efficiency* and  $\bar{a}$ -equal split guarantee trivially extends.

Assume a quantity of resources available at  $t$  can be transferred to the following/previous period by incurring in a linear and time-invariant iceberg cost  $\kappa \in (0, 1]$ .<sup>10</sup>

<sup>10</sup> If only transfers to later periods are allowed, the result of Theorem 2 immediately extends. Transfers to previous periods would equivalently emerge in a richer framework with production, when reducing capital investment.

If a portion  $\alpha \in [0, 1]$  of resources at  $t$  is transferred to period  $\tau$ , the part that becomes available for consumption at  $\tau$  is  $(1 - \kappa)^{|\tau - t|} \alpha$ . Let  $\mathcal{E}^\kappa$  be the domain of economies that generalize the ones described in Section 2 by allowing for transfers with strictly positive iceberg costs. An **economy** in the domain  $\mathcal{E}^\kappa$  is defined by the transfer cost and each agent's preferences:  $E \equiv (\kappa, \{\succsim_{i,t}\}_{\forall (i,t) \in I \times \mathbb{N}})$ .

**Theorem 6** *On the domain  $\mathcal{E}^\kappa$ , no rule satisfies Pareto efficiency and equal treatment of equals.*

*Proof* See Appendix A.

The intuition goes as follows: the “excess” of resources available to the agents living at periods with more favorable social environment cannot be equitably distributed to the benefit of infinitely many future generations living in less favorable environments. More specifically, since the transfer is costly (thus the delayed benefit of “saving” is below 1), it is not possible to sustain consumption bundles that exceed the unitary endowment (a similar argument is discussed in Asheim et al, 2010). The result is that, no matter how small the cost  $\kappa$  is, *efficiency* and *equal treatment of equals* remain non-compatible.

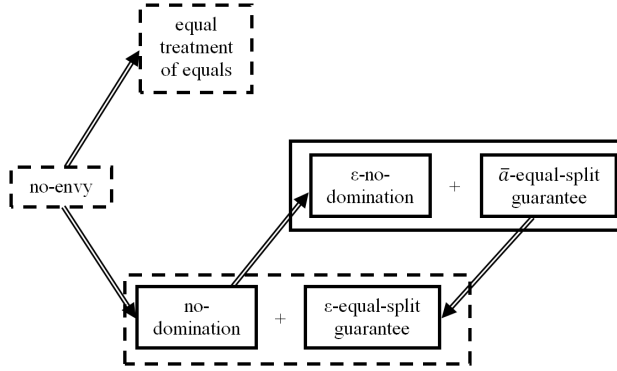
When the model instead allows for productive savings (or equivalently accumulation of capital), a different source of heterogeneity across generations may arise: the relative scarcity of goods at each period would not only depend on the composition of each generation (and their sequence over time), but also on the substitutability of goods at different times, determined by the specification of technology.<sup>11</sup>

## 5 Conclusions

We have studied how to equitably distribute a constant stream of goods in an overlapping generation framework, in which generations consist of (possibly heterogeneous) agents with ordinal and non-comparable preferences.

The results can be summarized by the following graph, where the equity axioms are compared. *No-envy* is stronger than *equal-treatment of equals* and *no-domination*; *no-domination* implies  $\varepsilon$ -*no-domination* while the  $\bar{a}$ -*equal-split guarantee* implies  $\varepsilon$ -*equal-split guarantee*. The tension between *efficiency* and equity is represented by the dashed boxes in which the axioms are, while the compatibility with *efficiency* is represented by continuous line boxes.

<sup>11</sup> This resource scarcity effect of production and accumulation is discussed in Piacquadio (2014); it arises already in a model with one-period living representative agent and with linear and time-invariant technology. More positive results are obtained by Fleurbaey (2007), but rely on a one-commodity setting with constant productivity.



*Equal-treatment of equals* is not compatible with *efficiency* due to different conditions (co-living agents) that same-preference agents face when they live in different periods. This implies the clash between *efficiency* and *no-envy*.

It is possible to define *efficient* distribution of resources that satisfy *no-domination*; this is however incompatible with giving each agent a bundle that she considers as desirable as an arbitrarily small share of the equal division of resources (*efficiency*, *no-domination*, and the  $\varepsilon$ -*equal-split guarantee* are not compatible). This result also implies that *efficiency* is not together compatible with *no-domination* and the  $\bar{a}$ -*equal-split guarantee*.

We suggest that an egalitarian criterion for this economy should be constructed starting from the  $\bar{a}$ -*equal-split guarantee*. It has been shown that with *efficiency* it implies  $\varepsilon$ -*no-domination* and that the equal-split allocation that entails the most concern for avoiding domination (thus maximizing  $\varepsilon$ ) is the “age-independent” division of resources, which equally divides the available goods between young and old agents.

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## A Proofs

### Theorem 4

*Proof.* Let  $\varepsilon \in (0, 1]$ . Consider a two-goods economy with one agent per generation (each agent is thus identified by its generation).

We shall first show as a lemma that for each  $\bar{a} \in A^{es}$  with  $\bar{a} \gg 0$ , there exist an economy  $E$  for which efficiency and no-domination can not be combined with making each agent at least as well-off than the  $\varepsilon$  part of  $(\bar{c}, \bar{d})$ . Such economy requires a specific pattern of preferences for a finite number of periods. To proof the theorem, we then construct an economy with an infinite sequence of such preference patterns, one for each element of a dense grid on  $A^{es}$ .

**Lemma 1** Let  $\bar{a} \in A^{es}$  with  $\bar{a} \gg 0$ . On the domain  $\mathcal{E}$ , there exists no rule that satisfies efficiency, no-domination, and such that for each agent  $(i, t) \in I \times \mathbb{N}_{>0}$ ,  $a_i(t) \succ_{i,t} \varepsilon \bar{a}_i(t) = \varepsilon(\bar{c}, \bar{d})$  and for each agent  $i \in I$ ,  $a_i(0) \succ_{i,0} \varepsilon \bar{a}_i(0) = \varepsilon \bar{d}$ .

*Proof* There are two-goods and one agent per generation. Let  $\bar{a} \in A^{es}$  be an *equal-split allocation* such that  $\bar{a}(t) = (\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2) \gg 0$ .

**Step 1.** Let the preferences of agents  $t-1, t, t+1 \in \mathbb{N}_{>0}$  be represented by the following linear functions:<sup>12</sup>

$$\begin{aligned} U_{t-1}(c_{t-1}^1, c_{t-1}^2, d_{t-1}^1, d_{t-1}^2) &= \gamma(c_{t-1}^1 + c_{t-1}^2) + \zeta d_{t-1}^1 + d_{t-1}^2 \\ U_t(c_t^1, c_t^2, d_t^1, d_t^2) &= \delta c_t^1 + c_t^2 + d_t^1 + \delta d_t^2 \\ U_{t+1}(c_{t+1}^1, c_{t+1}^2, d_{t+1}^1, d_{t+1}^2) &= c_{t+1}^1 + \zeta c_{t+1}^2 + \gamma(d_{t+1}^1 + d_{t+1}^2) \end{aligned}$$

where  $0 < \gamma < \delta < \zeta < \frac{\varepsilon}{3} \min[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2] < 1$ . For each agent  $i = t-1, t, t+1$  we denote by  $U_i^{ES}$  the utility level achieved at the *equal-split bundle*. The  $\varepsilon$  *equal-split guarantee* requires that  $U_i(a_i) \geq \varepsilon U_i^{ES}$ .

**Part a).** The distribution of goods available at time  $t$  is represented in the Edgeworth box of Figure 1a, where agent  $t-1$ 's origin is the bottom-left corner and agent  $t$ 's origin is the top-right corner. The *equal-split bundle* is denoted  $ES$ ; its  $\varepsilon$  component is  $\varepsilon ES^d$  for consumption when old and  $\varepsilon ES^c$  for consumption when young. By *efficiency*, the contract curve for the goods to distribute at period  $t$  (among consumption when young of  $t$  and consumption when old of  $t-1$ ) is such that either  $d_{t-1}^1 = 1$  or  $d_{t-1}^2 = 0$  (this corner solution follows from the linearity of preferences with  $\delta < \zeta$ ).

When  $d_{t-1}^2 = 0$ , it is not possible to assign to agent  $t-1$  a bundle that satisfies the  $\varepsilon$  *equal-split guarantee*: the maximum utility when  $d_{t-1}^1 = c_{t-1}^1 = c_{t-1}^2 = 1$  and  $d_{t-1}^2 = 0$  is  $U_{t-1}^{\max} = 2\gamma + \zeta < \varepsilon \min[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2] \leq \varepsilon \bar{d}^2 < \varepsilon U_{t-1}^{ES}$ . This is represented in the graph by  $U_{t-1}^{\max}$ ; whereas, the indifference level satisfying the  $\varepsilon$  *equal-split guarantee*, when consumption when young is maximum ( $c_{t-1}^1 = c_{t-1}^2 = 1$ ), is  $\bar{U}_{t-1}^d$ ; since this is higher than the  $U_{t-1}^{\max}$ , the *efficient* allocations that guarantee to agent  $t-1$  the equity constraint lie on the segment  $\bar{F}O_t^c$ : where  $d_{t-1}^1 = 1$  and  $d_{t-1}^2 > 0$ .

**Part b).** The distribution of goods available at time  $t+1$  is represented in the Edgeworth box of Figure 1b, where agent  $t$ 's origin is the bottom-left corner and agent  $t+1$ 's origin is the top-right corner. The *equal-split bundle* is denoted  $ES$ ; its  $\varepsilon$  component is  $\varepsilon ES^d$  for consumption when old and  $\varepsilon ES^c$  for consumption when young. By *efficiency*, the contract curve for the goods to distribute at period  $t+1$  (among consumption when young of  $t+1$  and consumption when old of  $t$ ) is such that either  $c_{t+1}^1 = 0$  or  $c_{t+1}^2 = 1$  (this corner solution follows from the linearity of preferences with  $\delta < \zeta$ ).

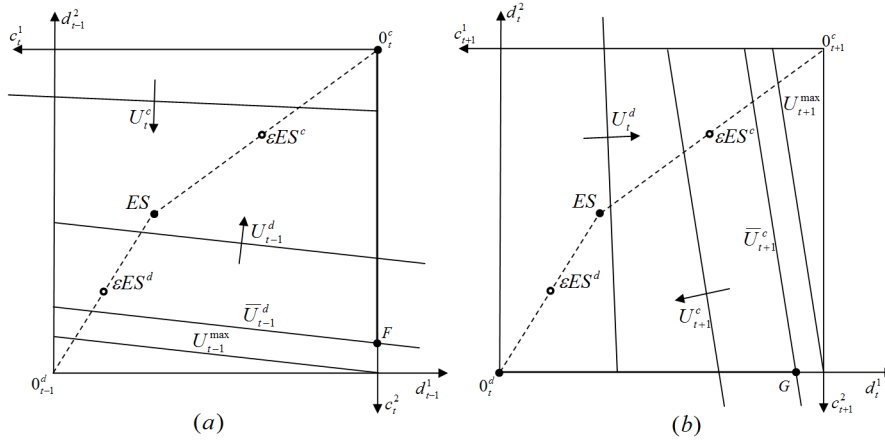
When  $c_{t+1}^1 = 0$ , it is not possible to assign to agent  $t+1$  a bundle that satisfies the  $\varepsilon$  *equal-split guarantee*: the maximum utility when  $c_{t+1}^2 = d_{t+1}^1 = d_{t+1}^2 = 1$  and  $c_{t+1}^1 = 0$  is  $U_{t+1}^{\max} = 2\gamma + \zeta < \varepsilon \min[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2] \leq \varepsilon \bar{c}^2 < \varepsilon U_{t+1}^{ES}$ . This is represented in the graph by  $U_{t+1}^{\max}$ ; whereas, the indifference level satisfying the  $\varepsilon$  *equal-split guarantee*, when consumption when old is maximum ( $d_{t+1}^1 = d_{t+1}^2 = 1$ ), is  $\bar{U}_{t+1}^c$ ; since this is higher than the  $U_{t+1}^{\max}$ , the *efficient* allocations that guarantee to agent  $t+1$  the equity constraint lie on the segment  $0_t^d \bar{G}$ : where  $c_{t+1}^1 > 0$  and  $c_{t+1}^2 = 1$ .

Summing up, the lifetime consumption of agent  $t$ ,  $a(t)$ , satisfies  $c_t^1 = 0$ ,  $c_t^2 < 1$ ,  $d_t^1 < 1$ , and  $d_t^2 = 0$ .

**Step 2.** Let the preferences of agents  $\tau-1, \tau, \tau+1, \tau+2 \in \mathbb{N}_{>0}$  with  $\tau = t+4$  be represented by the following utilities:

$$\begin{aligned} U_{\tau-1}(c_{\tau-1}^1, c_{\tau-1}^2, d_{\tau-1}^1, d_{\tau-1}^2) &= c_{\tau-1}^1 + c_{\tau-1}^2 + d_{\tau-1}^1 + \delta d_{\tau-1}^2 \\ U_{\tau}(c_{\tau}^1, c_{\tau}^2, d_{\tau}^1, d_{\tau}^2) &= c_{\tau}^1 + \zeta c_{\tau}^2 + \gamma(d_{\tau}^1 + \delta d_{\tau}^2) \\ U_{\tau+1}(c_{\tau+1}^1, c_{\tau+1}^2, d_{\tau+1}^1, d_{\tau+1}^2) &= \gamma(c_{\tau+1}^1 + \zeta c_{\tau+1}^2) + \zeta d_{\tau+1}^1 + d_{\tau+1}^2 \\ U_{\tau+2}(c_{\tau+2}^1, c_{\tau+2}^2, d_{\tau+2}^1, d_{\tau+2}^2) &= \delta c_{\tau+2}^1 + c_{\tau+2}^2 + d_{\tau+2}^1 + d_{\tau+2}^2 \end{aligned}$$

<sup>12</sup> The linearity assumption is without loss of generality: the result holds true when a second order term is added.



**Fig. 1** Determining the allocation of agent  $t$ .

where  $0 < \gamma < \delta < \zeta < \frac{\varepsilon}{3} \min[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2] < 1$ . For each agent  $i = \tau - 1, \tau, \tau + 1, \tau + 2$ ,  $U_i^{ES}$  denotes the utility level at the *equal-split bundle*; then, the  $\varepsilon$  equal-split condition requires that  $U_i(a_i) \geq \varepsilon U_i^{ES}$ .

**Part a).** The distribution of goods available at time  $\tau$  is represented in the Edgeworth box of Figure 2a, where agent  $\tau - 1$ 's origin is the bottom-left corner and agent  $\tau$ 's origin is the top-right corner. The *equal-split bundle* is denoted  $ES$ ; its  $\varepsilon$  component is  $\varepsilon ES^d$  for consumption when old and  $\varepsilon ES^c$  for consumption when young. By *efficiency*, the contract curve for the goods to distribute at period  $\tau$  (among consumption when young of  $\tau$  and consumption when old of  $\tau - 1$ ) is such that either  $c_\tau^1 = 0$  or  $c_\tau^2 = 1$  (this corner solution follows from the linearity of preferences with  $\delta < \zeta$ ).

When  $c_\tau^1 = 0$ , it is not possible to assign to agent  $\tau$  a bundle that satisfies the  $\varepsilon$  *equal-split guarantee*: the maximum utility when  $c_\tau^2 = d_\tau^1 = d_\tau^2 = 1$  and  $c_\tau^1 = 0$  is  $U_\tau^{\max} = \gamma(1 + \delta) + \zeta < \varepsilon \min[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2] \leq \varepsilon \bar{c}^1 < \varepsilon U_\tau^{ES}$ . This is represented in the graph by  $U_\tau^{\max}$ ; whereas, the indifference level satisfying the  $\varepsilon$  *equal-split guarantee*, when consumption when old is maximum ( $d_\tau^1 = d_\tau^2 = 1$ ), is  $\bar{U}_\tau^c$ ; since this is higher than the  $U_\tau^{\max}$ , the *efficient* allocations that guarantee to agent  $\tau$  the equity constraint lie on the segment  $0_{\tau-1}^d G'$ : where  $c_\tau^1 > 0$  and  $c_\tau^2 = 1$ .

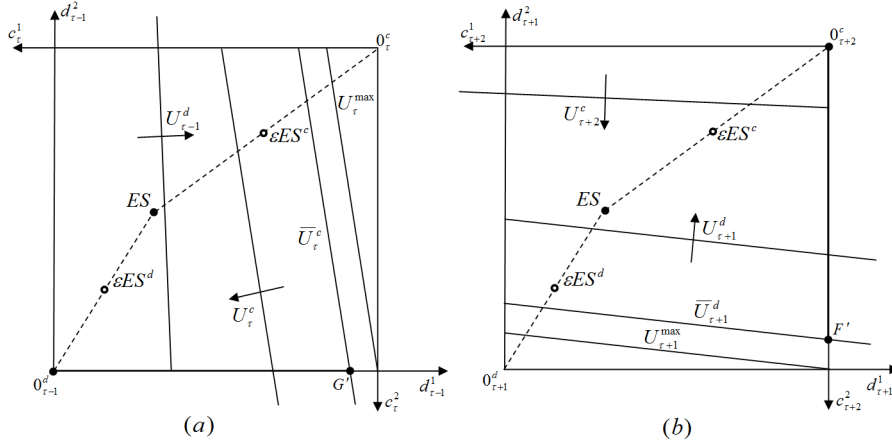
**Part b).** The distribution of goods available at time  $\tau + 2$  is represented in the Edgeworth box of Figure 2b, where agent  $\tau + 1$ 's origin is the bottom-left corner and agent  $\tau + 2$ 's origin is the top-right corner. The *equal-split bundle* is denoted  $ES$ ; its  $\varepsilon$  component is  $\varepsilon ES^d$  for consumption when old and  $\varepsilon ES^c$  for consumption when young. By *efficiency*, the contract curve for the goods to distribute at period  $\tau + 2$  (among consumption when young of  $\tau + 2$  and consumption when old of  $\tau + 1$ ) is such that either  $d_{\tau+1}^1 = 1$  or  $d_{\tau+1}^2 = 0$  (this corner solution follows from the linearity of preferences with  $\delta < \zeta$ ).

When  $d_{\tau+1}^2 = 0$ , it is not possible to assign to agent  $\tau + 1$  a bundle that satisfies the  $\varepsilon$  *equal-split guarantee*: the maximum utility when  $d_{\tau+1}^1 = c_{\tau+1}^1 = c_{\tau+1}^2 = 1$  and  $d_{\tau+1}^2 = 0$  is  $U_{\tau+1}^{\max} = \gamma(1 + \zeta) + \zeta < \varepsilon \min[\bar{c}^1, \bar{c}^2, \bar{d}^1, \bar{d}^2] \leq \varepsilon \bar{d}^2 < \varepsilon U_{\tau+1}^{ES}$ . This is represented in the graph by  $U_{\tau+1}^{\max}$ ; whereas, the indifference level satisfying the  $\varepsilon$  *equal-split guarantee* when consumption when young is maximum ( $c_{\tau+1}^1 = c_{\tau+1}^2 = 1$ ) is  $\bar{U}_{\tau+1}^d$ ; since this is higher than the  $U_{\tau+1}^{\max}$ , the *efficient* allocations that guarantee to agent  $\tau + 1$  the equity constraint lie on the segment  $F'0_{\tau+2}^c$ :  $d_{\tau+1}^1 = 1$  and  $d_{\tau+1}^2 > 0$ .

**Step 3.** The *efficient* distribution of resources available at  $\tau + 1$  requires that either:

- i)  $c_{\tau+1}^1 = 0$  (and  $d_\tau^1 = 1$ ); or
- ii)  $c_{\tau+1}^2 = 1$  (and  $d_\tau^2 = 0$ ).

From Step 1,  $a(t)$  satisfies  $c_t^1 = 0$ ,  $c_t^2 < 1$ ,  $d_t^1 < 1$ , and  $d_t^2 = 0$ .



**Fig. 2** Determining the allocations of agents  $\tau$  and  $\tau + 1$ .

From Step 2, in case *i*),  $a(\tau)$  is such that  $c_\tau^1 > 0$ ,  $c_\tau^2 = 1$ ,  $d_\tau^1 = 1$  and  $d_\tau^2 \geq 0$ : thus,  $a(\tau)$  dominates  $a(t)$ .

In case *ii*),  $a(\tau + 1)$  is such that  $c_{\tau+1}^1 \geq 0$ ,  $c_{\tau+1}^2 = 1$ ,  $d_{\tau+1}^1 = 1$ , and  $d_{\tau+1}^2 > 0$ : thus,  $a(\tau + 1)$  dominates  $a(t)$ .  $\square$

In two dimensions, each allocation in  $A^{es}$  can be mapped into a square with the proportion of each resource assigned to consumption when young on the two axes. We construct a dense grid over this square. First consider the center of the square, i.e.  $\bar{a} \in A^{es}$  such that  $(\bar{c}^1, \bar{c}^2) = (\frac{1}{2}, \frac{1}{2})$ . Second, by imaginary folding the square, once vertically and once horizontally, 4 smaller (overlapping) squares are obtained; the center of each is such that  $(\bar{c}^1, \bar{c}^2) = (a, b)$  with  $a, b \in \{\frac{1}{4}, \frac{3}{4}\}$ . Folding these squares again, once vertically and once horizontally, 16 smaller (overlapping) squares are obtained; the center of each is such that  $(\bar{c}^1, \bar{c}^2) = (a, b)$  with  $a, b \in \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}$ . Repeating this folding operation, a cell-centered coarsening method is described.

Let  $A^{es}(\nu)$  be the set of equal-split bundles that are a center of the squares obtained after  $\nu$  folding iterations.<sup>13</sup> For each  $\nu \geq 1$ , we can order the elements of  $A^{es}(\nu)$  lexicographically (good 1 first): for each  $\bar{a}, \bar{a}' \in A^{es}(\nu)$ ,  $\bar{a}$  is ranked before  $\bar{a}'$  if  $\bar{c}^1 < \bar{c}'^1$  or if  $\bar{c}^1 = \bar{c}'^1$  and  $\bar{c}^2 < \bar{c}'^2$ . Let  $\text{lex}(\bar{a}; \nu)$  denote the rank of  $\bar{a}$  in the set  $A^{es}(\nu)$ . Then each  $\bar{a}$  in the grid is uniquely identified by a function  $k$  that associates to it a number  $n \in \mathbb{N}_{>0}$  such that  $k(\bar{a}) = 1$  for  $\nu = 0$  and  $k(\bar{a}) = \frac{4^\nu - 1}{4 - 1} + \text{lex}(\bar{a}; \nu)$  for  $\nu \geq 1$ .

Let economy  $E \in \mathcal{E}$  be such that for each  $n \in \mathbb{N}_{>0}$  and the corresponding  $\bar{a} = k^{-1}(n)$ , preferences of agents  $\theta - 1, \theta, \theta + 1 \in \mathbb{N}_{>0}$  with  $\theta = 2 + 10n$  are as the ones of  $t - 1, t, t + 1$  in Step 1 of the above Lemma and preferences of agents  $\theta + 5, \theta + 6, \theta + 7, \theta + 8 \in \mathbb{N}_{>0}$  are as the ones of  $\tau - 1, \tau, \tau + 1, \tau + 2$  in Step 2 of the Lemma (note that these preferences depend on the choice of  $\bar{a}$  through the parameters  $\gamma, \delta, \zeta$  that depend on it).

Assume by contradiction that there exists an equal-split allocation  $\bar{a} \in A^{es}$  with  $\bar{a} \gg 0$  that satisfies the axioms for economy  $E$ . Since the grid is dense and preferences are continuous, there exists an  $n \in \mathbb{N}_{>0}$  such that for agents  $\theta - 1, \theta, \theta + 1, \theta + 5, \theta + 6, \theta + 7, \theta + 8 \in \mathbb{N}_{>0}$  with  $\theta = 2 + 10n$  the clash of the Lemma holds.  $\square$

## Theorem 6

*Proof* We construct an economy  $E \in \mathcal{E}^*$  for which no allocation satisfies *efficiency* and *equal treatment of equals*. Let  $L = 2$  and  $\kappa \in (0, 1]$ . Agents are of two kinds,  $\alpha$  and  $\beta$ , with preferences

<sup>13</sup> For example,  $A^{es}(2) \equiv \{\bar{a} \in A^{es} \mid (\bar{c}^1, \bar{c}^2) = (a, b) \text{ with } a, b \in \{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\}\}$ .

represented by the following functions, with  $\gamma > 1$  and  $\zeta \in (0, 1)$ :

$$\begin{aligned} U_\alpha(c^1, c^2, d^1, d^2) &= \zeta(\gamma c^1 + c^2) + \gamma d^1 + d^2 \\ U_\beta(c^1, c^2, d^1, d^2) &= \zeta(c^1 + \gamma c^2) + d^1 + \gamma d^2 \end{aligned}$$

Let generations  $t \in \{1, 2, 3, 4, 5\}$  consist of an  $\alpha$  and a  $\beta$  agent; let generations  $t \in \{7, 9, 11, \dots\}$  consist of two type  $\alpha$  agents and generations  $t \in \{8, 10, 12, \dots\}$  consist of two type  $\beta$  agents.

By *equal treatment of equals*, the  $\alpha$  agents of generations  $t \in \{7, 9, 11, \dots\}$  should achieve the same utility level, say  $\bar{U}_\alpha$ . Similarly, the  $\beta$  agents of generations  $t \in \{8, 10, 12, \dots\}$  should achieve the same utility level, say  $\bar{U}_\beta$ . The largest such utilities that can be feasibly assigned to these agents need to satisfy both the following constraints:  $\bar{U}_\alpha \leq \frac{1+\gamma}{2} + \zeta\left(\frac{1+\gamma}{2} - \bar{U}_\beta\right)$  and  $\bar{U}_\beta \leq \frac{1+\gamma}{2} + \zeta\left(\frac{1+\gamma}{2} - \bar{U}_\alpha\right)$ . When generations  $t \in \{1, 2, 3, 4, 5\}$  are *efficiently* assigned only the goods that cannot be assigned to other generations (generation 0 and generation 6), at least one agent ( $\alpha$  or  $\beta$ ) of these generations achieves a utility level larger than  $U^{\min} \equiv \gamma \frac{\zeta + \zeta^2 + \zeta^3 + \zeta^4}{1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4}$ . Clearly, when  $\zeta$  is close to 1 and  $\gamma$  is sufficiently large,  $U^{\min}$  is larger than the utility level agents of generations  $t \geq 7$  can equally achieve, i.e.  $\gamma \frac{\zeta + \zeta^2 + \zeta^3 + \zeta^4}{1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4} > \frac{1+\gamma}{2}$ .  $\square$

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