Working Paper 18/2014

The effect of small intervention costs on the optimal extraction of dividends and renewable resources in a jump-diffusion model

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The CREE Centre acknowledges financial support from The Research Council of Norway, University of Oslo and user partners.
ISBN: 978-82-7988-188-9
ISSN: 1892-9680
http://www.cree.uio.no

The effect of small intervention costs on the optimal extraction of dividends and renewable resources in a jump-diffusion model*

(This version 2014–11–27.)

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Abstract: A risk-neutral agent optimizes extraction of dividends or renewable natural resources modelled by a jump-diffusion stock process, where the optimal strategy is characterized as the minimal intervention required to keep the stock process inside a given region. The introduction of a small fixed cost $\varkappa$ per intervention, is shown to induce a loss at worst of order $\varkappa^{2/3}$, corresponding to a minimal intervention size of order $\varkappa^{1/3}$, under suitable conditions; there are degenerate cases if purely discontinuous harvesting is optimal for the frictionless problem. If extraction is reversible, at cost between half and twice the extraction cost, the exponents are 1/2 and 1/4, agreeing with the effect of fixed costs in a consumption–portfolio optimization problem for a risk-averse agent.

Key words and phrases: Optimal stochastic control, resource extraction, dividend extraction, jump-diffusion model, transaction costs.

MSC (2010): Primary: 91B28, 91B76. Secondary: 49L20, 49K20, 49K21, 60G40, 60H10, 62L15, 90C39, 93E20.

JEL classification: Q20, D23, G11.

*While carrying out this research, the author has been associated with the Oslo Centre for Research on Environmentally friendly Energy (CREE) and the Centre for the Study of Equality, Social Organization (ESOP) at the University of Oslo. CREE and ESOP are supported by The Research Council of Norway. The author gratefully acknowledges the hospitality of the ETH Zürich and discussions with H. Mete Soner.

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1 Introduction: the problem, the main result in brief, and related literature

This paper treats a risk-neutral agent's optimal extraction problem from $d \in \mathbb{N}$ possibly interacting stock quantities, with fixed *transaction cost* (a.k.a. *intervention cost* or simply *cost*) $\varkappa > 0$ – or without, the case $\varkappa = 0$, henceforth the *frictionless* problem. The main focus is the effect of the cost; it incurs at any time an intervention is made, with the same amount regardless of how much is extracted at that point in time, and from which and how many of the stocks. For any piecewise-constant nondecreasing extraction path $\{E(t)\}_{t\geq 0}$ (left-continuous) taking values in $\mathbb{R}^d_+$, our objective is to maximize the performance $J^{\varkappa}_E$ defined to be

$$J^{\varkappa}_E = \mathbf{E} \sum_{\tau;\ E(\tau^+)\neq E(\tau)} e^{-\beta\tau}\Big[\mathrm{d}E_1(\{\tau\}) + \ldots + \mathrm{d}E_d(\{\tau\}) - \varkappa\Big] \cdot 1_{\{\tau<\infty\}} \tag{1}$$

For $\varkappa = 0$ we can drop the assumption of piecewise constantness, but we are going to assume that those approximate the supremum. E is extracted from a jump-diffusion process Y taking values in $\mathbb{R}^d$ and satisfying

$$\mathrm{d}Y(t) = \mu(Y(t))\,\mathrm{d}t + \sigma(Y(t))\,\mathrm{d}W(t) + \int \zeta(Y(t^-), \varpi)\,\tilde{N}(\mathrm{d}\varpi, \mathrm{d}t) - \mathrm{d}E(t); \qquad Y(0) = y \tag{2}$$
$$\text{though with coordinate-wise absorbing zero; from first hitting time on, } Y_i \equiv 0 \equiv \mathrm{d}E_i$$

where $\tilde{N}$ is a centered Poisson random measure-valued martingale with Lévy measure ν and W is an $\mathbb{R}^{d_W}$-valued standard Brownian motion; σ takes $\mathbb{R}^{d\times d_W}$ matrix values, while the functions μ and ζ takes values in $\mathbb{R}^d$. For ϖ we do for the moment merely assume some suitable index set. Further conditions will be given below, ensuring well-definedness of the problem and the value function, which we can now define – as well as the «admissible» strategies:

$$v_{\varkappa}(y) = \sup\{J^{\varkappa}_E;\ E \text{ admissible}\}, \qquad \text{where } E \text{ is } \textit{admissible} \text{ if } \forall\, i = 1, \ldots, d: \tag{3}$$
$$E_i \text{ predictable, left-continuous, nondecreasing s.t. } E_i(0) = 0 \text{ and a.s. } Y_i(t) \geq 0\, \forall t \geq 0 \tag{4}$$

where we take as understood that «a.s.» means for the probability law of the process Y starting at y with strategy E. Condition (10) below will state a mild additional regularity condition.

One main object of this paper is the «loss» from $\varkappa$, by which we mean the difference $v_0 - v_{\varkappa}$. For the irreversible problems, we shall treat those where we extract upon «hitting from below». For $\varkappa = 0$, that means a reflecting boundary (downwards / «inwards» for the irreversible problem); intervention could also take the form of hitting an intervention set from «above» (or «outside») and then take it further down. There are certainly problems where the latter would be optimal – if we have a single stock which for low enough levels is doomed to decrease monotonously to zero, then it would be exploited immediately once it is low enough. Such cases, however – or at least, the behaviour at such thresholds – are not within the scope of this paper.

There is a wide literature on optimal resource extraction. We mention a handful of references under Brownian driving noise: [12], [13], [26], and a series of papers by Alvarez and coauthors, e.g. [2] and the references therein. [7] considers the effect of jump uncertainty, which may be different from the effect of Brownian. From the dividend extraction point of view, Paulsen [21] solves a class of problems with transaction costs under Brownian driving noise, and obtains our Theorem 3.2. We refer to his bibliography for more on the frictionless problem. While most of these works use Brownian motion-driven stochastic differential equations, [16] relates optimal reflection problems to the Feller property, in a model for production capacity adjustments.

Outline of the paper: This section will summarize the main results and discuss the connection to the related model of consumption–portfolio optimization. The next section will state the major assumptions to stand throughout the paper, and give the standard dynamic programming argument. Then we will elaborate on the one-dimensional problem: section 3 covers the problem where there are no jumps out of the continuation region (i.e. one can solve the Hamilton–Jacobi–Bellman equation and patch); the more complicated problem where jumps play a more significant rôle, will be treated in section 4, and the multi-dimensional problem in section 5. We have already claimed that the results crucially depend upon the irreversibility, and section 6 characterizes the solution for reversible problems where both costs are small (and sufficently close to equal), for the one-dimensional problem without jumps.

The main result can be summarized as follows: under suitable conditions, which are satisfied for a wide range of such problems, the frictionless value function v_0 is concave with intervention region $\mathit{\Gamma}_0$ and the optimal strategy being reflection off its boundary – i.e. the minimal intervention to keep $Y(t^+) \in \mathit{\Gamma}_0$ for all times – while the solution for each sufficiently small cost $\varkappa > 0$ has the following properties: There exist sets $\mathit{\Xi}_\varkappa \subset \mathit{\Gamma}_\varkappa \subset \mathbb{R}^d_+$ such that it is optimal to intervene whenever outside the closure of $\mathit{\Gamma}_\varkappa$ (and on the boundary, provided first exit time is zero), and optimal post-intervention state is $\in \partial\mathit{\Xi}_\varkappa \setminus \partial\mathit{\Gamma}_\varkappa$ – and as $\varkappa \searrow 0$, we have under suitable regularity conditions, the following asymptotic behaviour:

- $\mathit{\Xi}_\varkappa$ collapses to $\partial\mathit{\Gamma}_0$; usually we have the latter contained, so that $\partial\mathit{\Gamma}_0 = \cap_{\varkappa>0}\mathit{\Xi}_\varkappa$.
- For the irreversible problem, the minimum intervention size is under widely applying conditions (to be stated) of order of magnitude $\varkappa^{1/3}$; more precisely, if we let $\hat{y}^\varkappa \in \partial\mathit{\Gamma}_\varkappa$, chosen to converge to some $\hat{y}^0 \in \partial\mathit{\Gamma}_0$, then with $\check{y}$ being the corresponding optimal post-intervention state, then

$$\frac{\varkappa}{||\hat{y}^\varkappa - \check{y}^\varkappa||^3} \text{ is bounded, and «usually» bounded away from 0} \tag{5}$$

  while the value loss is usually of order of magnitude $\varkappa^{2/3}$:

$$\text{for each } y,\ \varkappa \mapsto \frac{v_0(y) - v_\varkappa(y)}{\varkappa^{2/3}} \text{ is bounded, and «usually» bounded away from 0.} \tag{6}$$

  - We calculate a quite general upper bound (Theorem 4.1) and under mild additional regularity conditions the exact limiting values of (5) and (6) (Theorems 3.2 and 4.2) for $d = 1$. For $d > 1$ we establish (Theorem 5.1) lower and upper bounds. Thus we do in particular give sufficient conditions for this «usually» lower bound to apply, wherein (6) will be sub-proportional to the contribution of the *continuous part* $\tilde{E}^c$ of the optimal $\tilde{E}$ for the *frictionless* problem
  - This contribution vanishes in particular for the following class of counterexamples: when even the frictionless problem has in optimum only finitely many interventions on each bounded time-interval. If so, then even using the $\tilde{E}$ strategy also under cost $\varkappa$, gives a loss which by comparing to a geometric series, is no larger than $\left(1 - \sup_{Y(0)} \mathbf{E}[\exp(-\beta \cdot \inf\{t > 0;\ \tilde{E}(t) > \tilde{E}(0^+)\})]\right)^{-1}\varkappa$, and if that supremum is < 1 then $[v_0(y) - v_\varkappa(y)]/\varkappa$ converges. Counterexample 4.4 elaborates further on this.

- For the *reversible* problem, then as $\varkappa^+ + \varkappa^- \searrow 0$, the orders are $(\varkappa^+ + \varkappa^-)^{1/4}$ resp. $(\varkappa^+ + \varkappa^-)^{1/2}$. As this problem is not the main topic of interest to this paper, we only establish these orders in a restricted case of no jumps and with costs «not too unequal», in order to exhibit that the 1/3 resp. 2/3 orders follow from the irreversible nature.

The problem has similarities to (as well as some important differences from) consumption–portfolio optimization, where reversible investments are known to exhibit the 1/3 resp. 2/3 orders for *proportional* costs, and more recently shown to exhibit 1/4 resp. 1/2 orders for fixed. We shall therefore outline some relations to this problem type. There are works on the effect of small costs also outside this application, e.g. [14], [17], [20], and one cannot expect an optimal control problem to be «well-behaved in all manners» with respect to vanishing costs, [6]. We do however remark that our model is not only well-behaved, but even trivial under *proportional* costs: Proportional costs in (at ratio $\lambda \geq 0$) in (1) would at intervention time yield $\big[(1-\lambda)\cdot(\mathrm{d}E_1(\{\tau\}) + \ldots + \mathrm{d}E_d(\{\tau\})) - \varkappa\big]$, and we could factor out $(1-\lambda)$ and consider $\varkappa/(1-\lambda)$ in place of $\varkappa$. Up to this rescaling, proportional costs do not impact the choice of strategy, and yield a one-to-one value loss.

Relation to consumption–portfolio optimization. Consider the Merton-type consumption–portfolio optimization problem, to maximize the expected aggregated discounted utility flow from consumption, drawn from a (locally riskless) bank account; the agent can also invest in risky opportunities («stocks»), and the agent is small in the sense of no impact on stock prices or interest rates. In the frictionless Merton problem, the portfolio will be taken to be self-financing apart from consumption, i.e. wealth changes will be only due to changes in market value as well as withdrawal for consumption. There is a vast literature on the problem with transaction costs – Cadenillas' survey [4] has a bibliography of 82 references before end 2000. The problem with *proportional* transaction costs has been extensively studied at least since Kamin [11] in the mid '70s. Among mathematicians the case of constant relative risk aversion utility is often referred to as the Davis/Norman problem, their work [5] characterizing the solution in terms of a ($\mathrm{d}t$-) singular local time reflection (with jumps, the «reflection» also has a discrete component catching jumps out of the continuation region, [8]. In Shreve's appendix to his and Soner's work [24], the effect of a small proportional transaction cost $\varkappa$ is found to be of order $\varkappa^{2/3}$ for the loss and $\varkappa^{1/3}$ for the width of the continuation region; further references on this property include [9], [10], [22], [23], [25] – although, the loss order is $\propto \varkappa^1$ when the agent only holds the risky asset (as proportional costs in our problem). For *fixed* transaction costs, references include [15], [19] and [1], the latter giving the effect to be a loss of order $\varkappa^{1/2}$ and interventions of size order $\varkappa^{1/4}$.

The problem of this paper is not precisely the same as the typical consumption–investment optimization problem. Let us outline some similarities and differences:

- The Merton-type problem assumes a small investor; while the value of an investment opportunity will ultimately evolve with nonlinear dependency of its extent (a firm might want to expand until the next dollar is equally well invested somewhere else), while the risky position of a small investor evolves proportionally to scale.

  In contrast, the optimal extraction of a stock of fish is akin to a large investor who owns an entire firm; the relative growth rate will typically depend on the stock level, just as the relative growth rate of a firm depends on whether it has already expanded past the most profitable activity, and, say, entered the nearly-unprofitable market segments.

- The extraction problem is thus akin to the one of extracting dividends from a firm (which does not trade in its own stock). However, we consider irreversible extraction, like if there is no way to fund expansion but through «organic growth». This is of course a limiting case of infinite issuance costs.

- In this paper, we assume risk-neutrality; future work will address risk-averse preferences. Under risk-neutrality the frictionless Merton problem under geometric stock price processes makes no sense, as the risk-neutral agent would borrow infinitely to buy a stock with positive expected excess drift; however one can consider risk-neutral agents under nonlinear stock dynamics with frictions. [25] and [22] make such assumptions on dynamics, but require strict concavity on direct utility which in their case imply concave *indirect* utility (i.e value function). Under fixed transaction costs, there is no reason to expect the latter concavity. It fails in the model of this paper, as well as in the model of [1].

- Although this paper assumes there is no bank account, we can easily introduce one for the sake of the comparison since utility is linear; rather than consuming immediately, we can assume that a unit extracted today, is added to an account X from which we consume a fixed fraction δ per time unit, so that $\mathrm{d}X = (r-\delta)X\,\mathrm{d}t + (Y(t^+) - Y(t))$ where r is the interest rate. Then a unit consumed initially will contribute as $1/(\beta + \delta - r)$, provided the denominator is positive; thus the difference to the dividend–consumption optimization problem is merely that consumption is fixed at rate $\delta X\,\mathrm{d}t$ rather than optimized. This fictitious bank account does however not give rise to a behaviour like in the Davis/Norman problem, where the stock exposure is kept within percentage levels of liquidation value; in this paper it will turn out to be only the value of the risky stock that matters.

Our irreversible problem gives the same impact of small *fixed* transaction costs, as the way small *proportional* costs do in the above problems. We shall see that this is due to the irreversibility. The relationship between these models, is the reason why this paper keeps some notational similarity with said references, using the «Y» letter for the resource stock process as that is risky. A future work will incorporate a risklessly evolving state «X» as well.

2 Global assumptions, and dynamic programming

Throughout the paper, we shall work on – and notationally suppress – the stochastic basis of a filtered probability space where the filtration is right-continuous and the initial sigma-algebra is complete in the sense that it contains all subsets of its null sets. On this we define Y by (2) with the entities as described in the text between that formula and (3). We use the following notational conventions:

- Y is right-continuous in between discontinuities in E which affects the process from the right-hand limit on; thus at a stopping time τ, $Y(\tau^+) - Y(\tau)$ – if nonzero – is the intervention while $Y(\tau) - Y(\tau^-)$ is the jump from the driving Poisson noise, if such one occurs at τ.

- The $||\cdot||$ denotes Euclidean norm in $\mathbb{R}^d$. For real numbers, $\{\,\cdot\,\}_+$ denotes positive part $\max\{0,\,\cdot\,\}$, applied before taking powers, so that e.g. $\{-1\}_+^2 = 0^2$ and not 1.

- We shall by abuse of notation suppress the ϖ in ζ and write «$\zeta(y)$».
- We notationally suppress the «$1_{\{\tau<\infty\}}$» occurring in (1), taking as understood that the event where an intervention does not occur in finite time, contributes zero to performance.
- $\mathcal{A}$ and $\mathcal{J}$ denotes the following operators, whenever well-defined:

$$\mathcal{A}g = -\beta g + (\nabla g)\mu + \frac{1}{2}\mathrm{tr}\big[\sigma\sigma^\top \nabla(\nabla g)^\top\big] + \mathcal{J}g \tag{7}$$

  everything evaluated at y, where $\mathcal{J}$ is the non-local operator associated with the jumps:

$$\mathcal{J}g(y) = \int \Big[g(y+\zeta(y,\varpi)) - g(y) - \nabla g(y)\,\zeta(y,\varpi)\Big]\,\nu(\mathrm{d}\varpi) \tag{8}$$

- By the *continuation region* we mean some open Γ such that it is strictly suboptimal to intervene whenever $y \in \Gamma$, and strictly suboptimal not to do so if y is in the interior of the complement. (We will mention it specifically when there is need to distinguish action or non-action on the boundary.) As we shall below assume that coordinates at zero are removed from the model, we shall denote $\partial\Gamma$ relative to the open first orthant.
- «Harvesting» or «extraction» means increasing E, considered irreversible except when stated.
- By abuse of terminology, we allow for the term «coefficients» of the differential equation to mean – depending on context – the functions μ, and either σ or σ^2, and then either ζ or the $y \mapsto \mathcal{J}g(y)$.

Let us state – and motivate – some conditions assumed as standing assumptions.

2.1 Assumption. Throughout the paper, the following hold unless otherwise explicitely stated.

(a) ϖ takes values in some suitable measurable index space, where $\zeta(y,\varpi)$ is a measurable function, ≥ -1 for all $y \geq 0$ and ν-a.e. ϖ, and satisfies

$$\sup_y y \mapsto \int ||\zeta|| \vee ||\zeta||^2 \,\mathrm{d}\nu < \infty. \tag{9}$$

We have already assumed that $Y_i \geq 0$, but the possibility of jumps past zero at certain states could *mandate* an immediate harvesting at any cost. Assuming $\zeta \geq -1$ avoids this, and keeps value nonnegative as one can always refrain from harvesting.

(b) On each compact cube $[1/s, s]^d$, we have $y \mapsto \mu$, $y \mapsto |\sigma|$ and $y \mapsto \mathcal{J}g(y)$ (each sublinear $g \in \mathsf{C}^3$) are all Lipschitz and of at most linear growth.
This ensures existence and uniqueness up until first hitting time of zero for a coordinate, and we have already ad hoc have assumed that 0 does trap each coordinate. Remark though, the deviation when referring to the Feller classification prior to Proposition 3.1.

(c) We assume the restriction to Markov step functions does not affect the supremum:

$$\begin{aligned} &\text{for } \varkappa > 0: \quad v_\varkappa = \max\,\{J_E^\varkappa;\ E \text{ admissible, Markov, piecewise constant}\} \\ &\text{for } \varkappa = 0: \quad v_0 = \ \sup\,\{J_E^\varkappa;\ E \text{ admissible, Markov, piecewise constant, } \varkappa > 0\}. \end{aligned} \tag{10}$$

When $\varkappa = 0$, we will get a solution characterized by reflection, but we assume that step controls can get arbitrary close. (This in particular uses the assumption that Y_i does not by itself cross zero.)

(d) $y_i \mapsto v_\varkappa(y)$ is strictly increasing for each small enough $\varkappa > 0$. (This obviously holds automatically if there are no jumps.)

(e) The initial state $Y(0)$ is coordinate-wise positive, and for each sufficiently small $\varkappa > 0$, there is some $s > 0$ such that whenever $Y_i < s$ it is never optimal to itervene *downwards.* (Otherwise, we would remove this coordinate from the model.)

(f) The frictionless irreversible problem has a continuation region $\mathit{\Gamma}_0 \ni 0$ that is convex and is not the entire positive orthant.

The dynamic programming argument. To the problem we associate the Hamilton–Jacobi–Bellman equation $\mathcal{A}g = 0$ (henceforth the «HJB equation») and the quasi-variational inequality

$$0 = \begin{cases} \max\Big\{\mathcal{A}g,\ 1 - \max_i \partial g/\partial y_i\Big\}, & \varkappa = 0 \\ \max\Big\{\mathcal{A}g(y),\ \sup_{y';\, y_i' < y_i \forall i} \{g(y') - g(y) + \sum_i (y_i - y_i') - \varkappa\}\Big\}, & \varkappa > 0 \end{cases} \tag{QVI}$$

(henceforth the «QVI»), where the operators are given by (7) and (8). We shall make the appropriate smoothness conditions for the QVI to hold for the true value function $v = v_\varkappa$ in the classical sense (without invoking the *viscosity* sense of a solution), at least locally. The argument is standard and can be found in e.g. [18]; we repeat it merely briefly for $\varkappa > 0$, for smooth enough nonnegative functions g to ensure the validity of the Dynkin formula and thus classical solutions:

$$\mathbf{E}\int_{\tau_n}^{\tau_{n+1}} e^{-\beta t}\mathcal{A}g(Y(t))\,\mathrm{d}t = \mathbf{E}\big[e^{-\beta\tau_{n+1}}g(Y(\tau_{n+1})) - e^{-\beta\tau_n}g(Y(\tau_n^+))\big] \tag{11}$$

holds for any two sufficiently integrable stopping times $\tau_n < \tau_{n+1}$ with no intervention in (τ_n, τ_{n+1}) (whether this be optimal or not). Passing through localizing sequences – truncate by $\bar\tau = \inf\{t \in [0,T];\ |\mathcal{A}g(Y(t))| + |\ln||Y(t)||| \geq T\}$ – we can assume that these stopping times are intervention times for a given strategy. Thus if $\liminf_n \mathbf{E}[e^{-\beta\tau_n}g(Y(\tau_n^+))] \geq 0$, then summing up and rearranging yields

$$g(y) + \mathbf{E}\int_0^{\bar\tau} e^{-\beta t}\mathcal{A}g(Y(t))\,\mathrm{d}t \geq \mathbf{E}\sum_{\substack{\text{intervention}\\ \text{times } \tau_n \leq \bar\tau}} e^{-\beta\tau_n}\left[g(Y(\tau_n)) - g(Y(\tau_n^+))\right] \tag{12}$$

Now if $g(y) \geq \sup\{g(y') + \sum_i (y_i - y_i') - \varkappa\}$ for all y, then passing to the limit on the right-hand side we get something $\geq J_E^\varkappa$. If for a given strategy, $\mathcal{A}g \leq 0$ wherever Y spends positive time, then the left-hand side is $\leq g$. This yields $g \geq J_E^\varkappa$, thus $g \geq v_\varkappa$ if the QVI holds everywhere; in that case we will even get equality as long as some strategy $\hat{E}$ attains the sup part and, letting $T \to \infty$ through natural numbers for this particular strategy, $\mathbf{E}[e^{-\beta\tau_T \wedge \bar\tau}g(Y((\tau_T \wedge \bar\tau)^+))] = 0$. This latter condition of course holds if the post-intervention state is bounded.

The above argument is the usual verification theorem. The following is also easy, but not so widely used in optimal control, as it gives rise to a *suboptimality* inequality, which is reverse to what is usually needed. Let us use (12) to evaluate a function g against the performance $J_E^{\varkappa}$ of a *given* (possibly suboptimal) strategy. First, the right-hand side of (12) is $\leq J_E^{\varkappa}$ (and thus $\leq v_{\varkappa}$) for all $\varkappa$ which for all intervention times, are $\leq \sum_i \left[Y_i(\tau_n) - Y_i(\tau_n^+)\right] - \left[g(Y(\tau_n)) - g(Y(\tau_n^+))\right]$, assuming that we can commute limit and expectation (for example if the bracketed difference on the right-hand side of (12) is nonnegative). If furthermore $\mathcal{A}g(y) \geq 0$ – this need only hold *whereever Y spends positive time under strategy E* – then $g \leq J_E^{\varkappa} \leq v_{\varkappa}$. (Notice that here we need not g to be smooth outside the range where Y spends time, as we do not maximize over strategies.)

We shall also use this for *approximate* inequalities, as we are interested in bounding the welfare loss $v_0 - v_{\varkappa}$ from the cost $\varkappa$. Indeed, we shall see that for a wide range of problems, choosing g as a downscaling v_0/η will bound the loss by order $\varkappa^{2/3}$. The argument will make use assumptions on the shape of the value function, and we need to justify that these do indeed cover a «wide range». The next sections will elaborate on the single variable case.

3 A single resource stock, $d = 1$: the continuous case and the easiest jumps

The following will give a class of one-dimensional problems with regularity properties that enable us to calculate the impact of intervention costs. Let us first make some general considerations if the value function $v = v_{\varkappa}$ is sufficiently smooth and Y is a continuous diffusion (ν vanishes).

If we for a second suspend the ad hoc assumption of 0 being absorbing, and suppose that even without this condition imposed, the coefficients are such that 0 as a boundary point is natural, or exit (or killing) in the Feller classification, then (cf. [3, pp. 17–19] there exists a nondecreasing function G («*G*rowing») unique up to scaling, satisfying $\mathcal{A}G = 0 = G(0)$. There also exists a nonincreasing solution, but it is G that is the interesting contribution for what follows; provided it is first concave then convex, we can patch at the inflection point $\tilde{y}$, assume G scaled to $G'(\tilde{y}) = 1$ and try $G(y \wedge \tilde{y}) + \{y - \tilde{y}\}_+$ as a candidate for the value function. The following gives sufficient conditions, their proof indeed showing the C^2 part which might not be completely clear from their theorem *statement:*

3.1 Proposition (adapted and simplified from Alvarez and Koskela [2, Theorem 2.3])**.** *Suppose C^2 coefficients with $\nu \equiv 0$, and that $\mu(y) - \beta y$ increases from a nonnegative value (strictly positive if 0 is attainable) at $y = 0$ to a unique maximum at a finite $\bar{y}^0 \geq 0$ (> 0 if 0 is unattainable), and decreases from $\bar{y}^0$ and crosses zero.*

Then the optimal strategy for the frictionless problem is reflection downward at a single threshold $\tilde{y} \geq \bar{y}^0$, being the unique zero of G'', where $G \in \mathsf{C}^2$ is uniquely given as the nondecreasing function satisfying $\mathcal{A}G = 0$ scaled to $G'(\tilde{y}) = 1$. The value function v_0 is C^2 and

$$v_0(y) = G(y \wedge \tilde{y}) + \{y - \tilde{y}\}_+. \tag{13}$$

That these conditions also ensure that $\tilde{y}$ is in the decreasing range of $\mu - \beta y$, we see by differentiating $\mathcal{A}G$:

$$0 = (\mu' - \beta)G' + (\mu + \frac{1}{2}(\sigma^2)')G'' + \frac{1}{2}\sigma^2 G''' \geq \mu'(\tilde{y}) - \beta \tag{14}$$

G'' cannot vanish anywhere to the left of $\bar{y}^0$, as this would lead to $G'''(\tilde{y}) < 0$. And, we cannot have $G'''(\tilde{y}) = 0$ unless possibly $\tilde{y} = \bar{y}^0$ and $\mu''(\bar{y}^0) = \sigma(\bar{y}^0) = 0$. Also, G indeed *inflects* where $G'' = 0$; otherwise, if G' were nonincreasing and not constant beyond the zero for $\mu - \beta y$, we would have the contradiction $0 = \frac{1}{2}\sigma^2 G'' + \mu G' - \beta G < (yG' - G)\beta/y < 0$. So G changes from (strictly) concave to (strictly) convex, and that again means that G' does on both sides of $\tilde{y}$ attain all levels sufficiently close to, but greater than, $G'(\tilde{y})$. From this we we also obtain the solution for positive costs if v_0 is as implied by Proposition 3.1 – see [21] for more on this problem:

3.2 Theorem. *Suppose the conclusion of Proposition 3.1 holds and that G is strictly concave on $(0, \tilde{y}]$ and strictly convex on $[\tilde{y}, \infty)$. Then for each $\hat{y} > \tilde{y}$, define $\check{y}$ to be 0 if $G'(0^+) \le G'(\hat{y})$ and otherwise, to be the unique $\check{y} < \tilde{y}$ for which $G'(\check{y}) = G'(\hat{y})$. Then the function*

$$v(y) = \frac{G(y \wedge \hat{y})}{G'(\hat{y})} + \{y - \tilde{y}\}_+ \tag{15}$$

is optimal for that problem which has cost

$$\varkappa = \hat{y} - \check{y} - \frac{G(\hat{y}) - G(\check{y})}{G'(\hat{y})} \quad (> 0) \tag{16}$$

and it is optimal to intervene whenever the stock is $\ge \hat{y}$, and if so, down to $\check{y}$.

Notice that there is a one-to-one correspondence between $\check{y}$ and $\varkappa$ until $\varkappa$ grows so large that $\check{y}$ hits zero – which never happens if $G'(0^+) = +\infty$.

Proof. The construction is easy; graph G (as printed in [21]) and add the tangent at some $\hat{y}$ in the convex part, and one parallel line above, either tangent in the concave part or through the origin. Scale down, and verify that this scaled v function satisfies $v(y) \ge \sup_{y'<y}\{v(y') + (y - y') - \varkappa\}$ with equality for $y \ge \hat{y}$, where the supremum is attained for $\check{y}$. It also satisfies $\mathcal{A}v = 0$ on $(0, \hat{y})$. The only part of the QVI which is yet to verify, is $\mathcal{A}v \le 0$ for $y > \hat{y}$; there we have $\mathcal{A}v(y) = \mu(y) - \beta v(y) = \mu(y) - \beta y - \beta(v(\hat{y}) - \hat{y})$. As $\tilde{y}$ is already in the decreasing range for the right-hand side, it suffices to check its limit as $y \searrow \hat{y}$. There $\mathcal{A}v$ has a downward discontinuity from the downward jump in the double derivative: $\mathcal{A}v(y) \to -\sigma^2(\hat{y})G''(\hat{y})/2G'(\check{y})$. Thus we can perform the standard dynamic programming routine, with the Itô formula holding true even across $\hat{y}$ where v is merely C^1 – noting that the growth condition required to vanish $\mathbf{E}e^{-\beta T}v(Y(T))$ holds because v is dominated by v_0 which by assumption is linear above $\tilde{y}$. □

3.3 Remark. Note that the formulation assuming the *conclusion* of Proposition 3.1, not the hypothesis; the conclusion applies wider, including for processes with jumps not out of $[0, \tilde{y}]$, see [7, Theorem 3.3 and Section 4].

This function form will give the transaction cost asymptotics for the continuous case. There is really nothing deep to the mathematics: the $\varkappa^{2/3}$ order is essentially the fact that if $g \in \mathsf{C}^3$ is (strictly) increasing at zero, with g, g' and g'' all vanishing there, then $g'(s)/[g(s)]^{2/3} \to [9g'''(0)/2]^{1/3}$ as $s \to 0$. More generally, if we allow g''' a single simple discontinuity, namely at zero, with left and right third derivatives $\vartheta^{\pm} \ge 0$, and for each $s \ne 0$ define $a(s)$ as the closest opposite-sign point with $g'(a(s)) = g'(s)$, we obtain the limit[1]

[1] This is routine l'Hôpital manipulations as the proof of Proposition 3.4, but we get slightly different coefficients; in the final expression of (17) we get $\sqrt[3]{9/8}$ not the $\sqrt[3]{9/2}$ in the text, as $g(a) \equiv 0$ violates the assumption on a; for a symmetric function, (17) is like dividing by $(2g)^{2/3}$. Also, formula (21) yields a third coefficient, $\sqrt[3]{9/32}$.

$$\lim_{s\to 0}\frac{g'(s)}{(g(s)+g(a(s)))^{2/3}}=\frac{1}{2}\sqrt[3]{9\Theta} \tag{17}$$

where – here and in the following – we define

$$\Theta := \frac{4\vartheta^+\cdot\vartheta^-}{(\sqrt{\vartheta^+}+\sqrt{\vartheta^-})^2}\quad \text{though 0 if } \vartheta^+=\vartheta^-=0. \tag{18}$$

Furthermore, $\lim_{s\to 0^\pm} a' = -\sqrt{\vartheta^\pm/\vartheta^\mp}$ provided well-defined, in which case the ratio $-a/(s-a)$ of the «a» side of the gap to the total gap, $s-a$, tends to

$$\frac{\sqrt{\vartheta^+}}{\sqrt{\vartheta^+}+\sqrt{\vartheta^-}},\qquad \text{(undefined if } \vartheta^+=\vartheta^-=0). \tag{19}$$

We shall need these constants in the following proposition:

3.4 Proposition. *Fix some function G such that $G'(\tilde y)=1$ for an inflection point $\tilde y$; assume that $G\in \mathsf{C}^2$ everywhere, and furthermore $G\in\mathsf{C}^3$* around except possibly precisely at $\tilde y$ *at which it has a left-sided resp. third derivative ϑ^-, resp. ϑ^+, and so that G is strictly concave to the left and strictly convex to the right. For each $\varkappa>0$ sufficiently small, let the conditions* (16) *and $G'(\check y)=G'(\hat y)$ define $\check y$ near, but below $\tilde y$ and $\hat y$ near, but above $\tilde y$. Then as $\varkappa\searrow 0$,*

$$\frac{1-\frac{1}{G'(\check y)}}{(\tilde y-\check y)^2}\to\frac{\vartheta^-}{2},\qquad \frac{1-\frac{1}{G'(\hat y)}}{(\hat y-\tilde y)^2}\to\frac{\vartheta^+}{2},\qquad \textit{and}\qquad \frac{1-\frac{1}{G'(\hat y)}}{(\hat y-\check y)^2}\to\frac{\Theta}{8}, \tag{20}$$

$$\lim_{\varkappa\searrow 0}\frac{\varkappa}{(\hat y-\check y)^3}=\frac{\Theta}{12}\qquad \textit{and}\qquad \lim_{\varkappa\searrow 0}\frac{G(y)-\frac{G(y)}{G'(\hat y)}}{\varkappa^{2/3}}=\frac{G(y)}{4}\sqrt[3]{18\Theta} \tag{21}$$

If in addition $\vartheta^+ + \vartheta^- > 0$, then $(\tilde y-\check y)/(\hat y-\check y)$ converges to (19).

Proof. The claims follow from l'Hôpital's rule, considering $\check y$ and $G'(\hat y)\varkappa = G(\check y)-G(\hat y)+(\hat y-\check y)G'(\hat y)$ as functions of $\hat y$. For (21), a single application of l'Hôpital's rule yields $(G'(\hat y)-1)/\varkappa^{2/3}\to(3/2)\cdot\lim\varkappa^{1/3}/(\hat y-\check y)$, so the two formulae of (21) are equivalent; using that $G''(\check y)(\mathrm{d}\check y/\mathrm{d}\hat y)=G''(\hat y)$ so that $-\mathrm{d}\check y/\mathrm{d}\hat y\to\sqrt{\vartheta^+/\vartheta^-}$ (assuming convergence, for the moment), the leftmost limit becomes

$$\frac{1}{3}\lim\frac{G''(\hat y)}{(\hat y-\check y)(1-\mathrm{d}\check y/\mathrm{d}\hat y)}=\frac{\vartheta^+}{3(1+\sqrt{\vartheta^+/\vartheta^-})^2} \tag{22}$$

as it should. As $1-\mathrm{d}\check y/\mathrm{d}\hat y\geq 1$, then even if it diverges the formula still holds. The last limit of (20) follows likewise, and dividing the two first we can verify that (19) fits the claim. □

This settles the asymptotic effect of small costs for a wide range of problems with jumps not out of the continuation region:

3.5 Corollary. *Under the applicability of Theorem 3.2 with G being C^3 at $\tilde y$,* (21) *yields*

$$\lim_{\varkappa\searrow 0}\frac{\varkappa}{(\hat y-\check y)^3}=\frac{v_0'''(\tilde y^-)}{12}\qquad \textit{and}\qquad \lim_{\varkappa\searrow 0}\frac{v_0(y)-v_\varkappa(y)}{\varkappa^{2/3}}=\frac{v_0(y\wedge\tilde y)}{4}\sqrt[3]{18v_0'''(\tilde y^-)}\,. \tag{23}$$

The next section will recover this result as a special case.

3.6 Remark.

(a) Under the hypothesis of Theorem 3.2, we can differentiate the HJB equation to get a positive third derivative of $\Theta = \vartheta = 2(\beta - \mu'(\tilde{y}))/\sigma^2(\tilde{y})$ provided $\sigma^2(\tilde{y}) > 0$; should it vanish then $\tilde{y}$ must be $= \bar{y}^0 = \operatorname{argmax}\{\mu - \beta y\}$ as in the deterministic problem, and a second differentiation of the HJB equation yields $\Theta = \vartheta = -\mu''(\bar{y}^0)/\beta$ – and thus an example of a vanishing third derivative, in the special case where μ'' vanishes at $\bar{y}^0$.

(b) The limits (20) and (21) admit $\Theta = 0$ (yielding loss less than the $\varkappa^{2/3}$ order), which holds if one of the one-sided third derivatives vanish. In the next section, we shall patch the value function with a cubic, and the arguments would obviously break down if $\vartheta^+ = \vartheta^- = 0$. In view of Proposition 3.4 applying with a possible discontinuity in the third derivative, the reader can verify that there is no use attempting some $\vartheta^+ > 0$ to fix the case $\vartheta^- = 0$ in order to get a positive $\lim(v_0 - v_\varkappa)/\varkappa^{2/3}$; (20) and (21) shows that the limit vanishes.

(c) In occation of orders different from the cubic, once could copy the argument of Proposition 3.4 as follows. Suppose that for some power $r > 1$, $\big[v_0(y) - v_0(\tilde{y}) - (y - \tilde{y})\big](y - \tilde{y})^{-r}$ is bounded away from both 0 and ∞. Then so does $\big[G'(\check{y}) - 1\big]\varkappa^{1-1/r}$, by copying the argument for cost like in (16). But from $r = 4$ we can infer that the argument is not universal – in the reversible model, where generically the lowest nonzero order derivative turns out to be the fourth (to be established in Section 6), the intervention size order of $\varkappa^{1/4}$ is accompanied by loss order $\varkappa^{1/2}$ and not $1 - 1/4$. Note as well that $r = 4$ should not occur in the irreversible model, as a zero for the third derivative would be a minimum.

4 More general jumps and the $1/3$ resp. $2/3$ orders

We have found the transaction cost asymptotics for the generic continuous case, and also when the non-localness of $\mathcal{A}$ poses no problems for the patching of functions at the (frictionless) optimal reflection threshold $\tilde{y}$. Let us now consider cases where jump terms may complicate the analysis, namely when we can *jump to intervention;* then $v_\varkappa$ and v_0 will, bar exceptional cases, not coincide near (but below) the frictionless threshold. If we make the assumption that the value $v_\varkappa$ is still concave-then-convex (then affine!), then it is tempting to suggest an *approximate* value function from scaling down v_0 to the left of its inflection point $\tilde{y}$, and then add a convex correction term from $\tilde{y}$ on – although the first theorem will work the other way, by upscaling $v_\varkappa$. Note that the non-local part $\mathcal{J}$ does evaluate to something positive for convex functions. Let us first clarify a piece of notation: up til now we had no need to distinguish between *suggested* and *optimal* strategies. Let us from now on denote suggested threshold levels by y^* and y_* and the actual optimal ones by $\hat{y} = \hat{y}^\varkappa$ and $\check{y} = \check{y}^\varkappa$.

Obviously, if $v_\varkappa$ has a derivative which is everywhere positive and ultimately one, we can scale it up by some large enough η until the derivative always exceeds one, in which case $\eta v_\varkappa$ will be superoptimal (and, possibly introducing the viscosity concept, pass the HJB-based superoptimality test). Part (a) of the following says little more, but the $\eta - 1 = 1/v' - 1$ factor is what is used to get the (21)-esque upper bounds (26). We have:

4.1 Theorem. *Assume for each small enough $\varkappa > 0$ that it is optimal to intervene iff $y > \hat{y}$ ($\varkappa$-dependent, $= \hat{y}^\varkappa$) and if so down to $\check{y} = \check{y}^\varkappa$, and that the value function $v_\varkappa$ is C^1 on $[\check{y}, \hat{y}]$ with $v'_\varkappa \leq 1$ there and $= 1$ at the endpoints. Denote $\tilde{y}^\varkappa = \operatorname{argmin}_{[\check{y},\hat{y}]} v'_\varkappa$.*

(a) *For each* $\eta \in (1, 1/v'(\tilde{y}^{\varkappa}))$ *(nonempty!) let* $\check{y} < y_* < y^* < \hat{y}$ *such that* $v'(y^*) = v'(y_*) = 1/\eta$. *Put* $\varkappa_\eta = y^* - y_* + [v_\varkappa(y_*) - v_\varkappa(y^*)] \cdot \eta$. *Then*

$$v_{\varkappa_\eta} \le \eta v_\varkappa(y \wedge y^*) + \{y - y^*\}_+ \tag{24}$$

and as $\eta \mapsto \varkappa_\eta$ *attains the entire range* $(0, \varkappa)$*, we get by* $\eta \nearrow v'(\tilde{y}^{\varkappa})$ *that on* $[0, \tilde{y}^0 \wedge \tilde{y}^{\varkappa}]$*:*

$$v_0(y) - v_\varkappa(y) \le \Big[\frac{1}{v'_\varkappa(\tilde{y}^{\varkappa})} - 1\Big] v_\varkappa(y) \tag{25}$$

(b) *Suppose in addition that each* $v_\varkappa$ *is* C^2 *with Lipschitz second derivative on* $(\check{y}, \hat{y})$ *(e.g. if the coefficients are* C^1*), with a* $\varkappa$*-uniform Lipschitz constant* Θ*, then on* $[0, \liminf_\varkappa \tilde{y}^{\varkappa}]$*:*

$$\limsup \frac{v_0 - v_\varkappa}{(\hat{y} - \check{y})^2} \le \frac{\Theta}{8} v_0, \qquad \limsup \frac{\varkappa}{(\hat{y} - \check{y})^3} \le \frac{\Theta}{12}, \qquad \limsup \frac{v_0 - v_\varkappa}{\varkappa^{2/3}} \le \frac{\sqrt[3]{18\,\Theta}}{4} v_0 \tag{26}$$

Proof. For $\varkappa_\eta$, we verify that $\eta v_\varkappa(y \wedge y^*) + \{y - y^*\}_+$ is superoptimal; by construction it satisfies the second part of the QVI, and it equals $\eta v_\varkappa(y)$ plus some concave function. If second derivatives are Lipschitz, then (26) follows like in Proposition 3.4, possibly by passing through convergent subsequences. □

We can improve on (26) under additional conditions, and indeed calculate the limit of $\varkappa^{-2/3}[v_0 - v_\varkappa]$ – it turns out that we shall remove the contribution from jumps out of the continuation region from v_0 in the rightmost expression. We could have calculated this from scaling up $v_\varkappa$ and patching with a linear, but that would lead to conditions on both the problems with and without costs; rather we shall in Theorem 4.2 do with a set of conditions that can be verified for the frictionles problem. We shall construct approximate value functions v_* from v_0 and calculate $\mathcal{A}v_*$ more explicitly.

Corresponding to suggested thresholds $y^* > y_*$ we denote the corresponding strategy by E^*; the optimal for the matching $\varkappa$ will be denoted $\hat{E}$, though also $\tilde{E}$ for the frictionless problem. If the optimal trigger for the frictionless problem is $\tilde{y}$ – dropping any zero superscript – we shall patch v_0 with v_0+ some function q_* to the right of $\tilde{y}$. This function will q_* be chosen C^1, and ultimately linearly increasing beyond a patching point $\hat{y}^*$ which we will choose as either y^* (with derivative equal to 1) or as $y^* \vee \hat{y}$ – the latter choice is to ensure that considerations we make on $(0, \hat{y}^*)$ are valid wherever the controlled process spends positive time. We thus start with $q \in \mathsf{C}^1$, vanishing on $[0, \tilde{y}]$, with $q'(\tilde{y}) = q''(\tilde{y}) = 0$, and for each $\hat{y}^* \ge \tilde{y}$ construct $q_* \in \mathsf{C}^1$ as

$$q_*(y) = q(\tilde{y} \vee y \wedge \hat{y}^*) + q'(\hat{y}^*)\{y - \hat{y}^*\}_+ \tag{27}$$

Then q_* is convex if q is convex on $[\tilde{y}, \hat{y}^*]$, in particular with the cubic choice

$$q(y) = \frac{\vartheta}{6}\{y - \tilde{y}\}_+^3, \tag{28}$$

where ϑ will equal the left third derivative of v_0 at $\tilde{y}$. With this choice we have $q'(\hat{y}^*) = (\hat{y}^* - \tilde{y})^2 \cdot \vartheta/2$ and we will have $q_*/(\hat{y}^* - \tilde{y}) \searrow 0$ monotoneously as $\hat{y}^* \searrow \tilde{y}$. Now for any given – but small enough – $\eta > 1$ we choose $y_* < \tilde{y}$ and $y^* > \tilde{y}$ by $(v_0 + q)' = \eta$, and restrict the choice of $\hat{y}^*$ to $\hat{y}^* \geq y^*$. We then denote by $v_* = (v_0 + q_*)/\eta$, and this will be our candidate for (approximate) value function for the following cost:

$$\varkappa = y^* - y_* + v_*(y_*) - v_*(y^*). \tag{29}$$

From Proposition 3.4 we know the order of the $v_0 - v_*$ difference, and we want to bound the order of the $v_* - v_\varkappa$ difference using (12). We have for any E that spends no time on $[\hat{y}^*, \infty)$,

$$\frac{v_* - J_E}{(\hat{y}^* - \tilde{y})^2} + \mathbf{E}\int_0^\infty e^{-\beta t}\,\frac{\mathcal{A}v_*(Y(t))}{(\hat{y}^* - \tilde{y})^2}\,\mathrm{d}t \geq 0 \tag{30}$$

with equality for E^*. (This does not contradict the fact $v_\varkappa \geq J_E$, because changing strategy changes the path of Y.) We shall however show that in the limit as cost tends vanishes – letting $\eta \searrow 1$ – the expectation converges to a value that does not depend on the strategy, such that we will obtain $\lim(v_* - J_{E^*})/(y^* - \tilde{y})^2 \leq \lim(v_* - v_\varkappa)/(\hat{y}^* - \tilde{y})^2$ as well as $(\hat{y}^* - \tilde{y})/(y^* - \tilde{y}) \to 1$. This asymptotic optimality will then give us the transaction cost order as long as the third derivative is positive, under some regularity conditions; note that the regularity conditions on the coefficients, are merely to ensure that $y \mapsto \mathcal{A}v_*$ has Lipschitz derivative which vanishes at $\tilde{y}$ – we could just as well have made that (weaker) assumption ad hoc. Introduce first the continuous part of a control E

$$E^{\mathsf{c}}(t) = E(t) - \sum_{\tau<t;\ E(\tau^+)\neq E(\tau)} \Big[E(\tau^+) - E(\tau)\Big] \tag{31}$$

which equals (since we are in dimension one)

$$E(t^-) - \Big[E(0^+) - E(0)\Big] - \int_0^{t^-}\!\!\int \{Y(T) + \zeta(Y(T),\varpi) - \tilde{y}\}_+\, N(\mathrm{d}\varpi, \mathrm{d}T). \tag{32}$$

Part (a) of the following theorem is already proven for $\vartheta = 0$, see Theorem 4.1.

4.2 Theorem. *Suppose for the frictionless problem that the optimal solution is reflection at $\tilde{y}$ with value function $v_0 \in \mathsf{C}^1$, and that for each $s > 0$ the following hold:*

- *On the left: $v_0 \in \mathsf{C}^3$ with Lipschitz third derivative $\to \vartheta > 0$ as $y \nearrow \tilde{y}$.*
- *On the right: $\sup_{y>\tilde{y}+s} \mathcal{A}v_0(y) < 0$*
- *On the neighbourhood $(\tilde{y}-s, \tilde{y}+s)$: the coefficients μ, σ^2 and $y \mapsto \mathcal{J}g(y)$ are differentiable with Lipschitz derivative, each $g \in \mathsf{C}^3$ of at most linear growth.*

(a) Then with $\tilde{E}$ denoting the optimal control for the frictionless problem,

$$\lim_{\varkappa\searrow 0} \frac{v_0 - v_\varkappa}{\varkappa^{2/3}} = \frac{\sqrt[3]{18\,\vartheta}}{4}\,\mathbf{E}\int_0^\infty e^{-\beta t}\,\mathrm{d}\tilde{E}^{\mathsf{c}}(t). \tag{33}$$

(b) Furthermore, if $v_\varkappa'''(\tilde{y}^\varkappa) \to \vartheta$ – for which it is sufficient that either (i) $\sigma^2(\tilde{y}) > 0$ or (ii) $\mu(\tilde{y}) + (\sigma^2)'(\tilde{y}) > 0$ – we have

$$\lim_{\varkappa\searrow 0} \frac{\varkappa}{(\hat{y} - \check{y})^3} = \frac{\vartheta}{12}. \tag{34}$$

Proof. For each $\eta > 1$, define as above $v_* = (v_0 + q_*)/\eta$, with q_* given by $q = \{y - \tilde{y}\}_+^3 \cdot \vartheta/6$ and (27). We have (30) valid, with equality if $\hat{y}^* = y^*$.

Consider $\mathcal{A}v_*$. $\mathcal{A}v_0 + (\mathcal{A} - \mathcal{J})q_*$ vanishes identically on $(0, \tilde{y}]$ as well as its derivative, and since v_* is C^3 on a neighbourhood of $\tilde{y}$, we have Lipschitz-continuous differentiability; $\mathcal{A}v_0 + (\mathcal{A} - \mathcal{J})q_*$ is on $(\tilde{y}, \hat{y}^*)$ bounded by some $M(\hat{y}^* - \tilde{y})^2/2$. As no time is spend above $\hat{y}^*$, dominated convergence yields

$$\mathbf{E}\int_0^\infty e^{-\beta t}\,\frac{\mathcal{A}v_0(Y(t)) + (\mathcal{A} - \mathcal{J})q_*(Y(t))}{(\hat{y}^* - \tilde{y})^2}\,\mathrm{d}t \to 0 \tag{35}$$

Consider the remaining part $\mathcal{J}q_*$. By checking the cases separately, one verifies that

$$\left|\frac{q_*(y+\zeta) - q_*(y) - \zeta q_*'(y)}{(\hat{y}^* - \tilde{y})^2}\right| \le \frac{\vartheta}{2}\cdot\begin{cases}\{y+\zeta-\tilde{y}\}_+ & \text{when } y+\zeta \ge \tilde{y} \text{ or } y \le \tilde{y}\\ |y+\zeta-\tilde{y}| & \text{when } y+\zeta \le \tilde{y} \text{ and } y \in (\tilde{y}, \hat{y}^*].\end{cases} \tag{36}$$

Therefore, by dominated convergence, we get for both E^* and the optimal $\hat{E}$ (where in the latter case we tacitly use convergence of the latter, cf. (10)),

$$\mathbf{E}\int_0^\infty e^{-\beta t}\,\frac{\mathcal{A}v_*(Y(t))}{(\hat{y}^* - \tilde{y})^2}\,\mathrm{d}t \to \frac{\vartheta}{2}\mathbf{E}\int_0^\infty e^{-\beta t}\int\{\tilde{Y}(t^-) + \zeta(\tilde{Y}(t^-)) - \tilde{y}\}_+\,\mathrm{d}\nu\,\mathrm{d}t \tag{37}$$

where $\tilde{Y}$ is the process controlled optimally for the frictionless problem. Thus as stated above, this gives an asymptotic reverse of the inequality $J_E \le v_\varkappa$:

$$\lim_{\eta\searrow 1}\frac{\hat{y}^* - \tilde{y}}{y^* - \tilde{y}} = 1 \qquad\text{and}\qquad \lim_{\eta\searrow 1}\frac{v_\varkappa - J_E}{(y^* - \tilde{y})^2} = 0 \tag{38}$$

whereby for $y \le \tilde{y}$, Proposition 3.4 gives

$$\lim_{\eta\searrow 1}\frac{(v_0 - v_*) + (v_* - v_\varkappa)}{(y^* - \tilde{y})^2} = \frac{\vartheta}{2}\Big[v_0 - \mathbf{E}\int_0^\infty\int e^{-\beta t}\{\tilde{Y}(t^-) + \zeta(\tilde{Y}(t^-)) - \tilde{y}\}_+\,N(\mathrm{d}\varpi, \mathrm{d}t)\Big] \tag{39}$$

– the «$N(\mathrm{d}\varpi, \mathrm{d}t)$» formulation exhibiting this as the instantaneous reaction to any jump out of $[0, \tilde{y}]$ – and the limit of $\varkappa/(y^* - \tilde{y})^3$, which yields the loss order part of (33) since the loss is constant from $\tilde{y} \vee \hat{y}$ on.

To establish the order of the intervention size, we have that

$$\frac{\varkappa/(y^* - \tilde{y})^3}{\varkappa/(\hat{y} - \tilde{y})^3} = \frac{(v_*(y_*) - v_*(y^*) + y^* - y_*)/(y^* - \tilde{y})^3}{(v_\varkappa(\check{y}) - v_\varkappa(\hat{y}) + \hat{y} - \check{y})/(\hat{y}^* - \tilde{y})^3} \tag{40}$$

The Lipschitz coefficient yield a third derivative for $v_\varkappa$ at its inflection point $\tilde{y}^\varkappa$ – although we only need a piecewise one, a $\Theta_\varkappa$: (40) implies

$$\Big(\frac{\hat{y} - \tilde{y}}{y^* - \tilde{y}}\Big)^3 = \frac{\vartheta + (y^* - \tilde{y})\tilde{M}_0}{\Theta_\varkappa + (\hat{y} - \tilde{y})\tilde{M}_\varkappa} \tag{41}$$

for bounded $\tilde{M} = \tilde{M}_\varkappa$. Now up to $\hat{y}$ we have $\mathcal{A}[v_\varkappa - v_*]$ vanishing identically. If $\sigma(\tilde{y}) \ne 0$, divide by it and differentiate term by term (by coefficient regularity) and see that it must match in the limit. If $\sigma(\tilde{y}) = 0$, differentiate twice and observe that the third derivative does not vanish from the equation. (Any contribution from the Lévy integral is positive in the limit and can not cancel out $\mu + (\sigma^2)'$.) □

So, under the regularity conditions imposed (including the positive third derivative), then introducing possible (compensated!) jumps out of the continuation region reduces v_0 (as any compensated jumps do, by concavity), but also reduces the relative (percentwise) loss from the intervention cost.

4.3 Remark.

(a) The condition on $\mathcal{A}v_0$ for $y > \tilde{y}$, ensures that the frictionless continuation region is unique, and rules out regions to the right of $\tilde{y}$ where one is indifferent between intervening or not. The assumption simplifies away the need for additional considerations on such regions under positive cost.

(b) Under the assumption $\vartheta > 0$, we have $\eta - 1 = (y^* - \tilde{y})^2\vartheta/2$, and for $y > y^*$:

$$\begin{aligned} v_*(y) - v_*(y_*) &= y - y^* + \frac{1}{\eta}\Big[y^* - \tilde{y} + \frac{\vartheta}{6}(y^* - \tilde{y})^3 + v_0(\tilde{y}) - v_0(y_*)\Big] \\ &\approx y - y_* - 2(y^* - \tilde{y}) + \frac{2}{\eta}\Big[y^* - \tilde{y} + \frac{\vartheta}{6}(y^* - \tilde{y})^3\Big] \\ &= y - y_* - \varkappa \qquad \text{with} \quad \varkappa = \frac{4(\eta-1)^{3/2}}{3\eta}\sqrt{\frac{2}{\vartheta}} \end{aligned} \tag{42}$$

– this value of $\varkappa$ matches in the limit, so that $(\eta-1)/\varkappa^{2/3} \to \frac{1}{4}\sqrt[3]{18\vartheta}$ (cf. (21) and (33)). We can also rewrite (37) into $\mathbf{E}\int_0^\infty e^{-\beta t}\frac{\mathcal{A}v_*(Y(t))}{\eta-1}\,\mathrm{d}t \to \mathbf{E}\int_0^\infty \int e^{-\beta t}\{\tilde{Y}(t^-)+\zeta(\tilde{Y}(t^-))-\tilde{y}\}_+\,\mathrm{d}\tilde{N}$. Furthermore, we could choose $\hat{y}^* = y^*$, because if $\hat{y} > y^*$ then $(\mathcal{A} - \mathcal{J})v_*(y)$ would have a contribution $-\frac{1}{2}\sigma^2\vartheta \cdot (\hat{y} - \tilde{y})$ which would only help proving the candidate function v_* to be at least near-superoptimal.

The next section shall make such a «$\vartheta > 0$» assumption, sacrificing a slight bit of generality for this convenience.

(c) Theorem 4.2 gives $[v_0 - v_\varkappa]\varkappa^{-2/3} \to 0$ when Y does only leave $[0, \tilde{y}^0]$ through upwards jumps. Of course, this degeneracy is avoided when $\sigma(\tilde{y}^0) \geq 0$, but the presence of Brownian noise it not a necessary condition for the order of $2/3$ – in fact, Theorem 3.2 applies to the deterministic problem, wherein (by differentiating the HJB equation) $-\mu v_0'' = (\mu' - \beta)v_0'$ exhibiting twice continuous differentiability across the patching point $\bar{y}^0 = \operatorname{argmax}\{\mu - \beta y\}$, provided $\mu \in \mathsf{C}^1$. Thus the loss order of $\varkappa^{2/3}$ prevails when the «reflection» is a $\mathrm{d}t$-absolutely continuous negative drift. Already in the introduction we outlined examples where the $\varkappa^{2/3}$ order does not apply, but the following will elaborate further on such a case.

4.4 Counterexample (A piecewise-constant process of loss order $\varkappa^1$, with $v_0 \notin \mathsf{C}^2$)**.** Suppose that N is compound Poisson with only upwards jumps, expected jump amplitude $\bar{\zeta}(y)$ and state-dependent intensity $\ell(y)$, and that Y when uncontrolled is piecewise constant – so $\mu = \ell\bar{\zeta}$. The optimal threshold is determined by trading off a unit now against the geometric series of ratio $\mathbf{E}e^{-\beta\tau} = \ell/(\beta + \ell)$ (with $\bar{\zeta}(y)\mathbf{E}e^{-\beta\tau}$ as first term) – this leads again to $\bar{y}^0 = \operatorname{argmax}\{\mu - \beta y\}$. And not only the threshold but also the *value for states above it* is as the deterministic problem, namely $(y - \bar{y}^0) + \mu(y^0)/\beta$. But the analogy does not carry over to the HJB equation, which – differentiating twice – leads to the usually negative left-handed second derivative

$$\lim_{y \nearrow \bar{y}^0} v_0''(y) = \frac{\bar{\mu}''(\bar{y}^0)}{\beta + \ell(\bar{y}^0)} \tag{43}$$

Should it by coincidence vanish, then the third derivative also does. Otherwise the discontinuity of the second derivative violate the applicability of the previous theorems. To get to a simple problem for the loss order due to small positive costs, let us assume that the jumps are not only positive, but so that for all small enough $\varkappa > 0$ we will intervene *at next jump,* assuming we start at the optimal post-intervention trigger $\check{y} = \check{y}^{\varkappa}$. The value at $\check{y}$ is then $[\bar{\zeta}(\check{y}) - \varkappa] \cdot \ell(\check{y})/\beta = [\mu(\check{y}) - \varkappa\ell(\check{y})]/\beta$, and the first-order condition for $\check{y}$ to be optimal (once we intervene!) is again, that the derivative be equal to 1. Under the assumption $\mu''(\bar{y}^0) < 0$, we get the following *first-order* effects by differentiating $\mu' - \varkappa\ell' = \beta$:

$$-\frac{\mathrm{d}\check{y}^{\varkappa}}{\mathrm{d}\varkappa} = \frac{\ell'(\check{y}^{\varkappa})}{\varkappa\ell''(\check{y}^{\varkappa}) - \mu''(\check{y}^{\varkappa})} \to \frac{\ell'(\check{y}^{\varkappa})}{|\mu''(\check{y}^{\varkappa})|} \qquad \text{so that} \qquad -\frac{\partial v_{\varkappa}}{\partial \varkappa}(\bar{y}^0) \to \frac{\ell(\bar{y}^0)}{\beta}. \tag{44}$$

The envelop theorem from calculus does in fact apply here.

5 More than one stock variable

This section will present the multidimensional problem. Leaving rigor until Theorem 5.1, we will first briefly make some geometric considerations at an intuitive level for the purpose of finding a reasonably wide class of problems for which the hypothesis of Theorem 5.1 applies; in the very least it should apply for independent stocks which, viewed separately, behave like in the previous section.

Consider, nonrigorously, the frictionless problem for independent stocks. By linearity of the criterion, the value function takes the form $v_0(y) = \sum f_i(y_i)$ where f_i is the frictionless value function when all other stocks but $\#i$ are 0 (which still is assumed absorbing for each). Let us assume that each f_i is C^2, strictly increasing and strictly concave up to $\tilde{y}_i$ and affine from there on – as in the previous section. Then the continuation region is $\{y;\ y_i < \tilde{y}_i\ \forall i\}$ while v_0 is affine on the translated orthant $\{y;\ y_i \geq \tilde{y}_i\ \forall i\}$. Introducing positive cost $\varkappa$ (sufficiently small) – incurring each time we harvest, no matter whether from one or both stocks – the value does no longer separate out additively. Reasonably, we will still not harvest a low stock even if we choose to harvest the other. Call the thresholds $\bar{y}_i$ and specialize for simplicity of visualization to $d = 2$: if $y_1 > \bar{y}_1$ there are y_2 for which one will harvest from both, while y_1 will never be touched as long as $< \bar{y}_1$ – and vice versa. Then the boundary of the continuation region has a horizontal component (near the second axis) and a vertical (near the first), and as $\varkappa \searrow 0$ the remaining part of the boundary collapses to a point. The curve that consists of the optimal post-harvesting states, will exhibit similar behaviour. Under strict concavity, there will be only one single point in the continuation region where both the two partial first-derivatives of v_0 equal a given $\eta > 1$.

The geometric heuristics motivate that the below assumptions admit, arguably, a wide range of problems. We give the main result, dropping the analogue of Theorem 4.1 and focusing on the analogue of Theorem 4.2 (i.e. the «$\vartheta > 0$» case). We only obtain interval bounds, but by the above considerations for the independent stocks it is not hard to find special cases where the ϑ_* and ϑ^* of the upcoming assumption number (iii) do coincide, and the limits are pinned down exact.

5.1 Theorem. *Assume the following hold true:*

(i) The continuation region $\mathit{\Gamma}_0$ for the frictionless problem, is a convex set, and there exists an open s-strip $\mathit{\Sigma}$ (nonempty, $s > 0$) around $\partial\mathit{\Gamma}_0$ such that on $\mathit{\Sigma}$,

- *the coefficients* μ, $\sigma\sigma^\top$ *and* $y \mapsto \mathcal{J}g(y)$ *are differentiable with Lipschitz derivative, for each* $g \in \mathsf{C}^1$ *of at most linear growth and* $\in \mathsf{C}^3$*;*
- v_0 *coincides with a concave* $g \in \mathsf{C}^2(\mathbb{R}^d_+)$*, strictly concave on* $\Sigma \cap$ *closure*(Γ_0)*.*
- v_0 *coincides on closure*(Γ_0) *with some* $\mathsf{C}^3(\mathbb{R}^d_+)$ *function with bounded Lipschitz third derivatives.*

(ii) For each such s-strip Σ, $\sup\{\mathcal{A}v_0(y);\ y \notin \Gamma_0 \cup \Sigma\} < 0$.

(iii) There are $\vartheta^* \geq \vartheta_* > 0$ *(not depending on i nor y) such that for each y which* $\in \partial_i\Gamma_0$ *for precisely one i, we have the third partial derivative* $(\partial/\partial y_i)^3 v_0(y) \in [\vartheta_*, \vartheta^*]$*. There exist d-dimensional problems where* $\vartheta_* = \vartheta^*$*.*

Then

$$\begin{aligned}\limsup_{\varkappa\searrow 0} \frac{v_0(y) - v_\varkappa(y)}{\varkappa^{2/3}} &\leq \frac{\sqrt[3]{18\,\vartheta^*}}{4}\ \mathbf{E}\int_0^\infty e^{-\beta t}\sum_{i=1}^d \mathrm{d}\tilde{E}_i^{\mathsf{c}}(t)\\ \liminf_{\varkappa\searrow 0} \frac{v_0(y) - v_\varkappa(y)}{\varkappa^{2/3}} &\geq \frac{\sqrt[3]{18\,\vartheta_*}}{4}\ \mathbf{E}\int_0^\infty e^{-\beta t}\sum_{i=1}^d \mathrm{d}\tilde{E}_i^{\mathsf{c}}(t)\end{aligned} \tag{45}$$

Furthermore, if $(\partial/\partial y_i)^3[v_0 - v_\varkappa](y) \to 0$ *for each* $y \in \partial\Gamma_0$*, then let* $\hat{y} = \hat{y}^\varkappa$ *be on the boundary of the continuation region for each* $\varkappa$*, such that there is a single coordinate #i it is optimal to intervene in (same i for all* $\varkappa$*), and converging to some* $\hat{y}^0 \in \partial\Gamma_0$ *as* $\varkappa \to 0$*; then*

$$\frac{\vartheta_*}{12} \leq \liminf_{\varkappa\searrow 0} \frac{\varkappa}{(\hat{y} - \check{y})^3}, \qquad \limsup_{\varkappa\searrow 0} \frac{\varkappa}{(\hat{y} - \check{y})^3} \leq \frac{\vartheta^*}{12}. \tag{46}$$

where $\check{y}$ *is the optimal post-intervention state corresponding to* $\hat{y}$*.*

Notice that the last claim does applies for the distances in y_i-direction – in the limit, that only misses some corners (of dimension $< d-1$).

The rest of this section will prove this theorem along the lines of the arguments from the univariate setup – however, because of the positivity condition (iii), we can simplify somewhat compared to Theorem 4.2. We will first construct a candidate v_* for value function for some intervention cost κ which we choose – *state-dependent!* – to fit to v_*. This will lead to Proposition 5.2 below, from which the theorem easily follows.

Let $v_0 + q \in \mathsf{C}^3$ around $\partial\Gamma_0$ and such that $q = 0$ on Γ_0. For each fixed $\eta > 1$ define Γ^* to be those y inside the surfaces defined by $\partial q/\partial y_i = \eta - 1$; that is, the maximal connected set containing 0 and Γ_0 such that the partial derivatives of q are $< \eta - 1$. Likewise, define Ξ_* by $\{y \in \Gamma_0;\ \partial v_0/\partial y_i < \eta\}$. For each $y \notin \Gamma_0$, define $\Upsilon = \Upsilon^*(y) \in \partial\Xi_* \cap \Gamma_0$ by the argmax over $y_* \leq y$ of $v_0(y_*) - \eta\sum_i y_i^*$. Employ the same abuse of notation for the boundary of Ξ_* as for $\partial\Gamma_0$, and write ∂_i for the parts of the boundaries as follows: $\partial_{i_1,\dots,i_{d'}}\Xi_*$ and $\partial_{i_1,\dots,i_{d'}}\Gamma^*$ where, respectively, $\partial v_0/\partial y_{i_1} = \partial v_0/\partial y_{i_{d'}} = \eta$ and $\partial q/\partial y_{i_1} = \partial q/\partial y_{i_{d'}} = \eta - 1$. This somewhat lengthy description gives a function $v_* = (v_0 + q_*)/\eta$. By calculations as in Remark 4.3 (b) along the line through $y \notin \Gamma_0$ and $\Upsilon^*(y) \in \partial\Xi_*$, we can define $\theta = \theta(y)$ as the directional third derivative at the point where the line crosses $\partial\Gamma_0$ (to be made precise below) and

$$\kappa = \kappa(y) := \frac{4(\eta-1)^{3/2}}{3\eta}\sqrt{\frac{2}{\theta(y)}} \tag{47}$$

If a problem has this particular state-dependent intervention cost, then the approximation $v_*(y) - v_*(\Upsilon) \approx \sum_i [y_i - \Upsilon_i] - \kappa(y)$ will be good enough for a multidimensional analogue of Theorem 4.2. We give the result:

5.2 Proposition. *Suppose hypotheses* (i) *to* (iii) *of Theorem 5.1 hold true. For each* $\eta > 1$*:*

- *Define* Γ^*, Ξ_*, v_* *and* $\Upsilon = \Upsilon^*(y)$ *as above, and let* E^* *denote the strategy of intervening iff* $y \notin \Gamma^*$ *and if so to* Υ*.*
- *For each* $y \notin \Gamma_0$ *define* $\theta(y)$ *as* $g'''(\tilde{s})$ *for* $g(s) = v_0(\Upsilon + (y - \Upsilon)s/||y - \Upsilon||)$*, with* $\tilde{s}$ *such that* $\Upsilon + (y - \Upsilon)\tilde{s}/||y - \Upsilon|| \in \partial\Gamma_0$*.*
- *Define the* y*-dependent cost* $\kappa(y)$ *by* (47)*, denote by* v_κ *the value function for this cost.*

Then as $\eta \searrow 1$*, with* $\tilde{E}$ *being optimal for the frictionless problem,*

$$\lim_{\eta \searrow 1} \frac{v_*(y) - v_\kappa(y)}{\eta - 1} = \lim_{\eta \searrow 1} \frac{v_*(y) - J_{E^*}(y)}{\eta - 1} = \mathbf{E} \int_0^\infty e^{-\beta t} \sum_{i=1}^d \Big(\mathrm{d}\tilde{E}_i(t) - \mathrm{d}\tilde{E}_i^{\mathsf{c}}(t) \Big) \tag{48}$$

and for the loss $v_0 - v_\kappa$ *we have*

$$\lim_{\eta \searrow 1} \frac{v_0(y) - v_\kappa(y)}{\eta - 1} = \mathbf{E} \int_0^\infty e^{-\beta t} \sum_{i=1}^d \mathrm{d}\tilde{E}_i^{\mathsf{c}}(t) \tag{49}$$

In particular, this implies that $\lim_{\eta \searrow 1} \kappa^{-2/3} \big[v_0(y) - v_\kappa(y) \big]$ *exists and is positive and finite on* $\partial\Gamma_0$*, where* (49) *equals*

$$\frac{4}{\sqrt[3]{18\,\theta(y)}} \cdot \lim_{\eta \searrow 1} \frac{v_0(y) - v_\kappa(y)}{\kappa^{2/3}}. \tag{50}$$

We shall employ essentially the same proof as in Theorem 4.2:

Proof. First, we note that Υ and θ are indeed well-defined by (strict) concavity near $\partial\Gamma_0$, by part (i) of the hypothesis – with $\theta > 0$ by (iii). By (ii) there does not emerge any other component of the continuation region, for small enough costs.

$\mathcal{A}v_0 + (\mathcal{A} - \mathcal{J})q_*$ vanishes identically on Γ_0, and by Lipschitz-continuous differentiability we have $\mathcal{A}v_0 + (\mathcal{A} - \mathcal{J})q_*$ bounded by some constant times $\vartheta_*^{-1}\sqrt{\eta - 1}$ on $\Gamma^* \setminus \Gamma_0$. For strategies where we spend no time outside Γ^*, then by dominated convergence,

$$\mathbf{E} \int_0^\infty e^{-\beta t} \, \frac{\mathcal{A}v_0(Y(t)) + (\mathcal{A} - \mathcal{J})q_*(Y(t))}{\eta - 1} \, \mathrm{d}t \to 0 \tag{51}$$

(cf. (35)) and $\mathbf{E} \int_0^\infty e^{-\beta t} \, \frac{\mathcal{J}q_*(Y(t))}{\eta - 1} \, \mathrm{d}t$ also converges dominated (cf. (36)), and we get (cf. (37)) that $\mathbf{E} \int_0^\infty e^{-\beta t} \, \frac{\mathcal{A}v_*(Y(t))}{\eta - 1} \, \mathrm{d}t$ converges to the discrete part of the optimal harvest for the frictionless problem: Namely, whenever $\tilde{Y}$ jumps outside Γ_0 and is discretely harvested back, it is counted, the contribution being $e^{-\beta\tau} \sum_i \Big\{ \tilde{Y}_i(\tau) - \Upsilon_i^*(\tilde{Y}_i(\tau)) \Big\}$, where the braced terms are all nonnegative and the sum is by construction strictly positive for the contribution to be counted. And for the particular case of E^*, (12) yields that

$$\lim_{\eta \searrow 1} \mathbf{E} \int_0^\infty e^{-\beta t} \, \frac{\mathcal{A}v_*(Y^*(t))}{\eta - 1} \, \mathrm{d}t = \lim_{\eta \searrow 1} \frac{J_{E^*} - v_*}{\eta - 1} \tag{52}$$

Compare this with the optimal strategy $\hat{E}^\kappa$ for the y-dependent cost κ: Should we then spend time outside, Γ^*, then as we cross $\partial\Gamma^*$, the double derivative of v_* makes a downward jump, and should (51) fail, the limit will be nonpositive – cf. Remark 4.3 (b). Should we intervene inside Γ^*, then we improve upon the other part of the QVI. We therefore get instead of (52) – with the integrand now evaluated at the optimal path, which (cf. (10)) approximates the optimal frictionless path in the limit – that

$$\lim_{\eta\searrow 1} \mathbf{E}\int_0^\infty e^{-\beta t}\,\frac{\mathcal{A}v_*(\hat{Y}(t))}{\eta-1}\,\mathrm{d}t \geq \limsup_{\eta\searrow 1}\frac{J_{\hat{E}^\kappa}-v_*}{\eta-1} \tag{53}$$

so that $\limsup(v_\kappa - J_{E^*})/(\eta-1) \leq 0$. The opposite inequality is obvious. Subtracting from $(v_0 - v_*)/(\eta-1)$ yields (49). □

This nearly proves the first part of Theorem 5.1 – and indeed, that it admits generalizations to y-dependent costs that can be written as a function bounded away from zero and infinity, times a scaling that we send to zero. We can now complete the proof of the theorem:

Proof of Theorem 5.1. For the inequalities (45): For each $\varkappa$ define η by (47) except with ϑ^* or ϑ_* in place of θ. Consider then the κ corresponding to that η – so that choosing $\varkappa$ to be the $\inf_y$ resp. the $\sup_y$ of $\kappa(y)$, so that v_κ upper bounds resp. lower bounds $v_\varkappa$. Apply Proposition 5.2 to get (45).

For the inequalities (46) we can adapt the single-variable argument along curves: fix a continuous $\varkappa \mapsto \hat{y}^\varkappa$. For each $\varkappa$ choose η so that $\hat{y}^\varkappa \in \partial\Gamma^*$, and choose κ accordingly. Let $\bar{y}^* \in \partial\Gamma_0$ on the line connecting $\hat{y}$ and $\Upsilon^*(\hat{y})$ (the one constructed using v_*) and let $\bar{y} \in \partial\Gamma_0$ on the line connecting with the true optimal post-intervention state. Rather than (40), we consider

$$\frac{\kappa/(y^* - \bar{y}^*)^3}{\varkappa/(\hat{y} - \bar{y})^3} \tag{54}$$

and expand like in (40). Again we get a ratio of third derivatives – in the same direction, by construction. We have assumed those of $v_\varkappa$ to converge while we know that those of v_* do. Thus the ratio converges, and to 1. □

6 Reversible interventions yield orders of $1/4$ resp. $1/2$

The main focus of this paper is the optimization over irreversible harvesting, and we have established the order $\hat{y} - \check{y} \propto \varkappa^{1/3}$ and a value loss of twice the order, i.e. $\propto \varkappa^{2/3}$, coinciding with the orders in the Black–Scholes consumption–investment problem under *proportional* costs. For fixed costs in *that* problem, the orders are 1/4 and 1/2, cf. [1]. Therein, transactions are assumed reversible (at a cost, of same order of magnitude). Those orders turn out valid in our model as well, if we allow reversibility in the form of a (cost and a) decreasing component of E – e.g. one could imagine moving a stock of animals to an area of lower population, or for finance applications to issue capital. Thus the minimal intervention has the order of magnitude that Remark 3.6 (c) hints at, yet that argument does suggest wrong loss order.

So, let us modify the setup to a simple model of reversible inverventions in a one-dimensional stock. For cost $\varkappa^+ \geq 0$ for harvesting, and $\varkappa^- \geq 0$ for decreasing E, we extend the class of controls to step functions E that are not necessarily monotone – apart from this, we keep the objective as before. We assume $\varkappa^+$ and $\varkappa^-$ both positive or both zero. For the latter, frictionless, case, it is evidently optimal to keep Y at $\bar{y}^0 = \operatorname{argmax}\{\mu(y) - \beta y\}$ so that the value is $v_0(y) = y - \bar{y}^0 + \mu(\bar{y}^0)/\beta$ for all $y > 0$. For the case with positive costs, we can formulate a verification theorem which we later will apply:

6.1 Proposition. *Suppose there are three points with $0 < \breve{y} < \bar{y} < \hat{y} < \infty$, and $v \in \mathsf{C}^1$ such that $0 \leq v(y) \leq y - \bar{y}^0 + \mu(\bar{y})/\beta$. Assume that*

- *on $(0, \breve{y}]$, $v' = 1$ and $\mu(y) - \beta y \leq \mu(\breve{y}) - \beta\breve{y}$;*
- *on $[\hat{y}, \infty)$, $v' = 1$ and $\mu(y) - \beta y \leq \mu(\hat{y}) - \beta\hat{y}$;*
- *on $(\breve{y}, \hat{y})$, v is C^2 with bounded second derivative and solves $\mathcal{A}v = 0$;*
- *$v(y) - y$ is strictly increasing on $(\breve{y}, \bar{y})$ and strictly decreasing on $(\bar{y}, \hat{y})$.*

Then v is the value function for the problem with costs

$$\begin{aligned} \varkappa^+ &= v(\bar{y}) - v(\hat{y}) + \hat{y} - \bar{y} \qquad (> 0) \\ \varkappa^- &= v(\bar{y}) - v(\breve{y}) + \breve{y} - \bar{y} \qquad (> 0) \end{aligned} \tag{55}$$

and it is optimal to intervene iff $Y \notin (\breve{y}, \hat{y})$ and if so to $\bar{y}$.

Proof-sketch. The dynamic programming argument is standard, and we only fill in the detail that assumptions on μ ensures $\mathcal{A}v \leq 0$: Because $\mathcal{A}[y + a] = \mu(y) - \beta y - \beta a$, it suffices to evaluate $\mathcal{A}$ at the endpoints of the interval, where by the (possible) discontinuity in the second derivative, $\mathcal{A}v$ jumps from 0 to $-\frac{1}{2}\sigma^2 v'' \leq 0$ as we pass from the inside and out. □

Conversely, a C^1 value function v for a problem where the continuation region is a bounded interval $(\breve{y}, \hat{y})$ bounded away from zero, must necessarily have $v' = 1$ at the endpoints and also at $\bar{y} := \operatorname{argmax}\{v(y) - y\}$. For the case without jumps, we can for suitable costs and coefficients construct the solution as follows – notice that for the function Q, the value and derivative at the particular point $\bar{y}^0$ coincide with the frictionless case value:

6.2 Theorem. *Assume C^2 coefficients μ and σ^2, and no jumps ($\nu = 0$). Assume there is a unique $\bar{y}^0 = \operatorname{argmax}\{\mu(y) - \beta y\}$, and that $\mu''(\bar{y}^0) < 0 < \sigma^2(\bar{y}^0) \cdot (\mu(\bar{y}^0) - \beta\bar{y}^0)$. Fix $\rho \in (1/2, 2)$.*

(a) Then for all small enough $\hat{y} > \bar{y}^0$, there is a reversible problem with continuation region $(\breve{y}, \hat{y})$, where $\breve{y} := \bar{y}^0 - (\hat{y} - \bar{y}^0)/\rho$, and with costs are given by (55), where $\bar{y} \in (\breve{y}, \hat{y})$ maximizes $v(y) - y$, and the value function v is constructed as

$$v(y) = \frac{1}{\eta} Q(y) + \alpha P(y) \qquad \text{on } (\breve{y}, \hat{y}) \tag{56}$$

uniquely defined by the following:

$$\mathcal{A}Q = \mathcal{A}P = 0, \qquad Q(\bar{y}^0) = \frac{\mu(\bar{y}^0)}{\beta}, \qquad P(\bar{y}^0) = 0, \qquad Q'(\bar{y}^0) = P'(\bar{y}^0) = 1, \tag{57}$$

$$\frac{1}{\eta} = \frac{P'(\breve{y}) - P'(\hat{y})}{Q'(\hat{y})P'(\breve{y}) - Q'(\breve{y})P'(\hat{y})} \quad \in (0,1) \qquad \text{and} \qquad \alpha\eta = \frac{Q'(\hat{y}) - Q'(\breve{y})}{P'(\breve{y}) - P'(\hat{y})} \quad > 0. \tag{58}$$

(b) As $\hat{y} \searrow \bar{y}^0$, with $\breve{y} = \bar{y}^0 - (\hat{y} - \bar{y}^0)/\rho$ (so that $(\hat{y} - \bar{y}^0)/(\bar{y}^0 - \breve{y})$ is kept constant $= \rho$), we have the following limits, all strictly positive – where $\phi = Q''''(\bar{y}^0) = \frac{-\mu''(\bar{y}^0)}{\frac{1}{2}\sigma^2(\bar{y}^0)}$:

$$\frac{\varkappa^+}{(\hat{y} - \bar{y}^0)^4} \to \lim \frac{\varkappa^+}{(\hat{y} - \breve{y})^4} \cdot \Big(\frac{\rho + 1}{\rho}\Big)^4 = \frac{\varphi}{24} \cdot (2\rho - 1)^3 \rho^{-4} \tag{59}$$

$$\frac{\varkappa^-}{(\bar{y}^0 - \breve{y})^4} \to \lim \frac{\varkappa^-}{(\hat{y} - \breve{y})^4} \Big(\rho + 1\Big)^4 = \frac{\varphi}{24} \cdot (2\rho^{-1} - 1)^3 \rho^4 \tag{60}$$

$$\sup_{y \in (\breve{y}, \hat{y})} \left| \frac{v_0(y) - v(y)}{(\hat{y} - \breve{y})^2} - \frac{-\mu''(\bar{y}^0)}{6\beta} \cdot \frac{\rho^2 - \rho + 1}{(\rho + 1)^2} \right| \to 0, \tag{61}$$

$$\sup_{y \in (\breve{y}, \hat{y})} \left| \frac{v_0(y) - v(y)}{\sqrt{\varkappa^\pm}} - \frac{\rho^2 - \rho + 1}{\beta} \sqrt{\frac{1}{3} \cdot \frac{-\mu''(\bar{y}^0)\sigma^2(\bar{y}^0)}{\rho^{2\pm 1}(2 - \rho^{\mp 1})^3}} \right| \to 0 \qquad \textit{and} \tag{62}$$

$$\sup_{y \in (\breve{y}, \hat{y})} \left| \frac{v_0(y) - v(y)}{\sqrt{\varkappa^+ + \varkappa^-}} - \frac{\rho^2 - \rho + 1}{\beta} \sqrt{\frac{1}{3} \cdot \frac{-\mu''(\bar{y}^0)\sigma^2(\bar{y}^0)}{-\rho^4 + 14\rho^3 - 12\rho^2 + 14\rho - 1}} \right| \to 0 \tag{63}$$

(c) Conversely, consider a sequence of problems, whose respective continuation regions are $(\breve{y}, \hat{y}) \ni \bar{y}^0$, and each value function is C^2 outside $\{\breve{y}, \hat{y}\}$ and C^1 everywhere, and classically solves the QVI. Suppose that in the limit, $\hat{y} \searrow \bar{y}^0$ and $\breve{y} \nearrow \bar{y}^0$. Then

$$\frac{1}{2} \leq \liminf \frac{\hat{y} - \bar{y}^0}{\bar{y}^0 - \breve{y}}, \qquad \limsup \frac{\hat{y} - \bar{y}^0}{\bar{y}^0 - \breve{y}} \leq 2. \tag{64}$$

Proof. At $\bar{y}^0$ we have $Q'' = 0$, and from $(\mathcal{A}Q)' = 0$ also $Q''' = 0$. Differentiate once more to get $Q''''(\bar{y}^0) = -2\mu''(\bar{y}^0)/\sigma^2(\bar{y}^0) =: \varphi$ which by assumption is strictly positive. So near this point, $Q(y) - Q(\bar{y}^0) - (y - \bar{y}^0)$ is of order four and convex, while $P(y) - (y - \bar{y}^0)$ is concave and of quadratic order, $P''(\bar{y}^0) = -2\mu(\bar{y}^0)/\sigma^2(\bar{y}^0)$. Therefore, with $\alpha > 0$ and $\eta > 0$, $v(y) - y$ is approximately a W-shaped quartic, strictly concave near $\bar{y}^0$, and when α is low enough and η is sufficiently close to 1, we have at least two local minima for $v(y) - y$ and a maximum in between, call it $\bar{y}$ and notice that (55) then fits. With η and α as in (58), we have $v'(\breve{y}) = v'(\hat{y}) = 1$, and for convexity/concavity near (what we claim are) the local extrema, we must have η and α both positive, and as $|\hat{y} - \breve{y}| \searrow 0$ they tend to 1 resp. 0. To check that $1 - 1/\eta > 0$, we have $\eta - 1 = Q'(y) - 1 + \eta\alpha P'(y)$ for $y \in \{\breve{y}, \hat{y}\}$, and $Q'(y) - 1$ is of higher order than $\eta\alpha$.

We need to make sure that $\hat{y}$ and $\breve{y}$ are where appropriate; namely, on the opposite sides of $\bar{y}^0$, and also on the opposite sites of the third stationary point of the W-shaped $v(y) - y$. Making the coordinate change into $\xi = (y - \bar{y}^0)/(\hat{y} - \breve{y})$, with accents on ξ corresponding to those on y; viz., $\hat{\xi} = \rho/(1 + \rho)$ while $\breve{\xi} = -1/(1 + \rho)$. We now claim that $\bar{\xi} \to -(\hat{\xi} + \breve{\xi}) = (1 - \rho)/(1 + \rho)$. At the three zeroes for $v' - 1$ we have

$$Q' - 1 + \alpha\eta \cdot (P' - 1) = \eta - 1 - \alpha\eta = \frac{[Q'(\hat{y}) - 1][P'(\breve{y}) - 1] - [Q'(\breve{y}) - 1][P'(\hat{y}) - 1]}{P'(\breve{y}) - P'(\hat{y})} \tag{65}$$

Divide throughout by $(\hat{y} - \breve{y})^3$ and calculate limits, we obtain the (limiting) cubic equation $[\xi - \hat{\xi}] \cdot [\xi - \breve{\xi}] \cdot [\xi + (\hat{\xi} + \breve{\xi})] = 0$ with roots $\hat{\xi}$, $\breve{\xi}$ and $-(\hat{\xi} + \breve{\xi}) = \lim \bar{\xi}$. Somewhat surprising, the last root lies on the «smallest» side, $-(\hat{\xi} + \breve{\xi}) > 0$ if $\hat{\xi} < |\breve{\xi}|$, and it is between the two others if $\rho \in (1/2, 2)$ and not if $\rho \notin [1/2, 2]$. Requiring that v is convex around $\breve{y}$ and around $\hat{y}$ yields the same for ρ; for the latter, consider

$$\eta \frac{v''(\hat{y})}{(\hat{y}-\breve{y})^2} = \frac{Q''(\hat{y})}{(\hat{y}-\breve{y})^2} + \frac{\alpha\eta P''(\hat{y})}{(\hat{y}-\breve{y})^2} \to \frac{\varphi}{2}[\hat{\xi} - 1/3] \tag{66}$$

i.e. the $\hat{y} - \bar{y}^0$ interval should exceed 1/3 of $\hat{y} - \breve{y}$, i.e. half of the other part, $\rho > 2$. For $\breve{y}$ the argument is similar. This also proves part (c), as this construction is the only possible classical solution to the QVI around $\bar{y}^0$ assuming this lies in the continuation region. It also proves that for (fixed) $\rho \in (1/2, 2)$, v has the properties claimed, and we furthermore have

$$\frac{1-1/\eta}{(\hat{y}-\breve{y})^2} \to \lim \frac{\alpha\eta}{(\hat{y}-\breve{y})^2} = \frac{\varphi}{-6P''(\bar{y}^0)} \cdot \frac{\rho^2-\rho+1}{(\rho+1)^2} = \frac{-\mu''(\bar{y}^0)}{6\mu(\bar{y}^0)} \cdot \frac{\rho^2-\rho+1}{(\rho+1)^2} \tag{67}$$

and (as $\sup_{(\breve{y},\hat{y})} |P| \to 0$) we have for the maximum loss on the collapsing continuation region:

$$\sup_{y\in(\breve{y},\hat{y})} \frac{v_0(y)-v(y)}{(\hat{y}-\breve{y})^2} \to Q(\bar{y}^0) \lim \frac{1-1/\eta}{(\hat{y}-\breve{y})^2}. \tag{68}$$

To calculate the limits of $\varkappa^{\pm}/(\hat{y}-\breve{y})^4$, consider now $\eta\varkappa^+ = \int_{\bar{y}}^{\hat{y}} (1-v'(y))\eta \,\mathrm{d}y$, noting that $v'(\hat{y}) = v'(\bar{y}) = 1$; we have $\eta - \eta v'(y) = \eta - 1 - \alpha\eta - (Q'(y)-1) - \alpha\eta(P'(y)-1)$ and $\eta - 1 - \alpha\eta$ equals $Q' - 1 + \alpha\eta(P'-1)$ at any of the three stationary points for $v - y$. Choosing $\hat{y}$ for the upper part, and inserting for $\alpha\eta$ from (58), we get

$$\frac{\varkappa^+}{(\hat{y}-\breve{y})^4} = \frac{1/\eta}{\hat{y}-\breve{y}} \int_{\bar{y}}^{\hat{y}} \Big[\frac{Q'(\hat{y})-Q'(y)}{(\hat{y}-\breve{y})^3} + \frac{Q'(\hat{y})-Q'(\breve{y})}{(\hat{y}-\breve{y})^3} \cdot \frac{P'(\hat{y})-P'(y)}{P'(\breve{y})-P'(\hat{y})}\Big] \mathrm{d}y \tag{69}$$

$$\text{which} \quad \to \frac{\varphi}{6} \int_{\lim\bar{\xi}}^{\hat{\xi}} \Big\{\hat{\xi}^3 - \xi^3 - \Big[\hat{\xi}^3 - \breve{\xi}^3\Big](\hat{\xi} - \xi)\Big\} \mathrm{d}\xi \tag{70}$$

$$\text{which} \quad = \frac{\varphi}{24}\hat{\xi}^4(1-\lim\bar{\xi}/\hat{\xi})^2\Big\{(1+\lim\bar{\xi}/\hat{\xi})^2 - 2\Big[1+\breve{\xi}/\hat{\xi}\Big]\breve{\xi}/\hat{\xi}\Big\} = \frac{\varphi}{24} \cdot \frac{(2\rho-1)^3}{(\rho+1)^4} \tag{71}$$

– here is used that $\hat{\xi} - \breve{\xi} = 1$ so that $\hat{\xi}^3 - \breve{\xi}^3 = \hat{\xi}^2[1 + (\breve{\xi}/\hat{\xi}) + (\breve{\xi}/\hat{\xi})^2]$; furthermore, we have inserted for $\lim \bar{\xi}/\hat{\xi} = \rho^{-1} - 1$ and $\breve{\xi}/\hat{\xi} = -\rho^{-1}$. For $\varkappa^- = \int_{\bar{y}}^{\breve{y}} (v'-1)\,\mathrm{d}y = \int_{\breve{y}}^{\bar{y}} (1-v')\,\mathrm{d}y$, so we can switch rôles of $\hat{\xi}$ and $\breve{\xi}$ in (71), and insert for $\lim \bar{\xi}/\breve{\xi} = \rho - 1$ and $\hat{\xi}/\breve{\xi} = -\rho$ to get

$$\frac{\varkappa^-}{(\breve{y}-\breve{y})^4} \to \frac{\varphi}{24}\breve{\xi}^4(1-\lim\bar{\xi}/\breve{\xi})^2\Big\{(1+\lim\bar{\xi}/\breve{\xi})^2 - 2\Big[1+\hat{\xi}/\breve{\xi}\Big]\hat{\xi}/\breve{\xi}\Big\} = \frac{\varphi}{24} \cdot \frac{(2\rho^{-1}-1)^3}{(\rho^{-1}+1)^4} \tag{72}$$

The rest is algebraic manipulations. □

For the irreversible problem, the arguments for the continuous case would also go through for jumps not out of the continuation region. In this case though, where the continuation region collapses, those jumps must be such that post-jump states $y + \zeta(y)$ all lie between y and $\bar{y}^0$ – this at least to hold for all small enough neighbourhoods. Arguably, such cases seem artificial. However, the approach does admit jumps to zero at fixed intensity ν_0 (as then $\mathcal{J}g = -\nu_0 g + \nu_0 y g'$, only modifying β and μ in a way that maintains the derivative $\mu' - \beta$ unchanged).

7 Closing remarks

This paper has established a value loss from the transaction cost $\varkappa$ of order $\varkappa^{2/3}$ and a minimum intervention size of order $\varkappa^{1/3}$ for a wide range of problems, as well as counterexamples; the irreversibility yields different orders than reversible problems, for which the effect of small costs is akin to the one applicable for consumption–portfolio optimization.

The assumption of risk-neutral preferences distinguishes this paper from the common consumption–portfolio optimization in a Black–Scholes-type (linear) market. Preliminary considerations lead the author to conjecture that the effects of transaction costs do not crucially depend on neither the particular choice of preferences nor on the absence of a risk-free investment opportunity. This is work in progress.

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